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Superconnections and Index Theory

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Superconnections and Index Theory

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Superconnections and Index Theory

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This document presents a systematic investigation of the geometric index theory of Dirac operators coupled superconnections. A local version of the index theorem for Dirac operators coupled to superconnection is proved, and extended to families. An η -invariant is defined, and it is shown to satisfy an APS-like theorem. A geometric determinant line bundle with section, metric, and connection is associated to a family of Dirac operators coupled to superconnections, and its holonomy is calculated in terms of the η -invariant.

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Chapter 1

Introduction

To a large extent, index theory is the study of Dirac operators on manifolds, and the topological and geometric invariants associated to these.

In order to define Dirac operators, we need to review some of the geometry of Riemannian manifolds. Every Riemannian manifold X has a canonical Clifford algebra bundle associated with it, $\text{Cl}(X) \rightarrow X$. If, in addition, the manifold is spin then the choice of a spin structure gives a distinguished \mathbb{Z}_2 -graded $\text{Cl}(X)$ -module $\mathbb{S}(X) \rightarrow X$, called the bundle of spinors, in terms of which every other \mathbb{Z}_2 -graded $\text{Cl}(X)$ -module may be decomposed.¹ We denote the action of $\text{Cl}(X)$ by $c(\cdot)$.

We are now in a position to define Dirac operators. Let $V \rightarrow X$ be a complex and hermitian vector bundle, with connection ∇ . Then the *Dirac operator* associated to (V, ∇) is defined by the sequence

$$\mathcal{D}(\nabla) : \Gamma(\mathbb{S}(X) \otimes E) \xrightarrow{\nabla^{\mathbb{S}(X) \otimes V}} \Omega^\bullet(X; \mathbb{S}(X) \otimes E) \xrightarrow{c(\cdot)} \Gamma(\mathbb{S}(X) \otimes E),$$

¹We take spinors to be \mathbb{Z}_2 -graded throughout. Conventionally the spinors on odd dimensional manifolds $\mathbb{S}^0(X) \rightarrow X$ are taken to be ungraded. The \mathbb{Z}_2 -graded spinors on an odd dimensional manifold are defined to be $\mathbb{S}(X) = \mathbb{S}^0(X) \oplus \mathbb{S}^1(X)$, where the second copy is to be odd. Clifford multiplication by odd elements maps elements from the first copy to the second copy, and vice-versa; in particular, if vol is the volume form on X , then $i^{\dim X} \text{vol}$ is an odd endomorphism of $\mathbb{S}(X)$ that squares to one.

Here $\nabla^{\mathbb{S}(X) \otimes E}$ is the connection induced on $\mathbb{S}(X) \otimes E$ from the Levi-Civita connection on $\mathbb{S}(X)$ and ∇ on E . On compact manifolds Dirac operators are elliptic. If the connection ∇ is unitary, then the Dirac operator is self-adjoint. In this case the Dirac operator decomposes as

$$\mathcal{D}(\nabla) = \begin{pmatrix} & \not{D}(\nabla)^* \\ \not{D}(\nabla) & \end{pmatrix} \quad (1.1)$$

in terms of the grading on $\mathbb{S}(X) \otimes E$. The operator $\not{D}(\nabla)$ is *Fredholm*, and thus has a well-defined index associated to it:

$$\text{index } \not{D}(\nabla) = \dim \ker \not{D}(\nabla) - \dim \text{coker } \not{D}(\nabla).$$

The square of the Dirac operator is called the laplacian on E .

1.1 The Atiyah-Singer index theorem

The basic theorem for Dirac operators on manifolds is the Atiyah-Singer index theorem² [7].

Theorem 1.2. *Let X be a smooth, compact and spin Riemannian manifold. Let $V \rightarrow X$ be a complex and hermitian vector bundle, with unitary connection ∇ . Then*

$$\text{index } \not{D}(\nabla) = \hat{A}(X) \text{ch}(V)[X].$$

A large variety of naturally occurring elliptic operators on manifolds may be written as Dirac operators.

²It should be remarked that the Atiyah-Singer index theorem may be stated more generally [8], and applied to elliptic operators that are not expressible as Dirac operators coupled to connections.

Example 1.3 (The Euler Characteristic). Let X be a smooth and compact Riemannian manifold. The operator $d + d^*$ acting on $\Omega(X)$, graded by cohomological degree reduced mod 2, may be realised as a Dirac operator. In this case the index theorem asserts that

$$\text{index}(d + d^*) = \chi(X).$$

On the other hand, Hodge theory tells us that $H(X, \mathbb{R}) \cong \ker(d + d^*)$, so that

$$\text{index}(d + d^*) = \sum_{k=0}^{\dim X} (-1)^k \dim H^k(X, \mathbb{R}).$$

Thus we obtain the well known formula

$$\chi(X) = \sum_{k=0}^{\dim X} (-1)^k \dim H^k(X; \mathbb{R}).$$

Example 1.4 (The Hirzebruch signature theorem). Let X be a compact and spin Riemannian manifold with $\dim X = 4n$. Define the map

$$\sigma : \Omega^k(X) \rightarrow \Omega^{4n-k}(X)$$

defined on homogeneous differential forms by

$$\sigma : \omega \mapsto i^{|\omega|(|\omega|-1)} \star \omega,$$

where \star is the Hodge star operator. The map σ squares to 1, so that we may decompose $\Omega(X)$ by its ± 1 eigenspaces of σ : $\Omega(X) \cong \Omega^{+1}(X) \oplus \Omega^{-1}(X)$, and use this to grade $\Omega(X)$. Furthermore, \star anti-commutes with $d + d^*$, and so is a map $d + d^* : \Omega^{+1}(X) \rightarrow \Omega^{-1}(X)$. The index of $d + d^*$ is the signature of X , denoted by $\text{Sign } X$. The index theorem shows that

$$\text{Sign } X = L(X)[X],$$

where $L(X)$ is the L -genus of X .

Example 1.5 (The Riemann-Roch Theorem). Let X be a compact complex manifold with complex vector bundle V with connection ∇ . Let

$$\bar{\partial}_V : \Omega^{(0,i)}(X; V) \rightarrow \Omega^{(0,i+1)}(X; V)$$

be the twisted anti-holomorphic differential. The operator $\bar{\partial}_V + \bar{\partial}_V^*$ may be realised as a Dirac operator on $\Omega^{(0,\cdot)}(X; V)$ graded by degree reduced mod 2. The Hodge theorem shows that

$$\text{index}(\bar{\partial}_V + \bar{\partial}_V^*) = \sum_p (-1)^p \dim H^p(X; V).$$

On the other hand, the index theorem asserts

$$\text{index}(\bar{\partial}_V + \bar{\partial}_V^*) = \text{ch}(V) \text{Td}(X)[X].$$

Combining these one has the Riemann-Roch theorem

$$\sum_p (-1)^p \dim H^p(X; V) = \text{ch}(V) \text{Td}(X)[X].$$

The index theorem (Th. 1.2) is essentially topological: it equates the analytic index of the Dirac operator to a topological invariant, and forgets much of the geometry required to define the Dirac operator. One would like to refine this theorem to capture some of this geometry. The “topological side” of the index theorem has a natural refinement: there are canonical differential forms representing the characteristic classes, and these may be thought of as a “local version” of the topological index. It is, however, not immediately clear

what an appropriate local refinement of the analytic index might be. The answer to this question is supplied by the McKean-Singer formula [27]. One may form a heat operator associated to the Dirac operator:

$$H = \frac{\partial}{\partial t} + \mathcal{D}(\nabla)^2. \quad (1.6)$$

Associated to this is the heat semigroup:

$$e^{-t\mathcal{D}(\nabla)^2}. \quad (1.7)$$

The McKean-Singer formula states that the analytic index is computed by the supertrace (or \mathbb{Z}_2 -graded trace) of the heat semigroup³

$$\text{index } \not{D}(\nabla) = \text{Tr } e^{-t\mathcal{D}(\nabla)^2} = \text{Tr}_0 e^{-t\not{D}(\nabla)^* \not{D}(\nabla)} - \text{Tr}_0 e^{-t\not{D}(\nabla) \not{D}(\nabla)^*}. \quad (1.8)$$

The heat semigroup is a smoothing operator, and is thus represented by an integral kernel:

$$e^{-t\mathcal{D}^2(\nabla)}\phi = \int_X dy p_t(\cdot, y)\phi(y).$$

In terms of the integral kernel, the index is computed by

$$\text{index } \not{D}(\nabla) = \int_X dx \text{Tr } p_t(x, x).$$

In the light of this, it is reasonable to regard

$$\text{Tr } p_t(x, x)dx$$

as a local refinement of the analytic index. The local index theorem [29, 23, 1] equates these:

³Throughout this document, “Tr” will designate the supertrace, and “Tr₀” the ungraded trace.

Theorem 1.9. *Let X be a smooth, compact, spin and Riemannian manifold. Let $V \rightarrow X$ be a complex and hermitian vector bundle, with unitary connection ∇ . Then*

$$\lim_{t \rightarrow 0} \text{Tr } p_t(x, x) dx = (2\pi i)^{-\dim X/2} \left[\hat{A}(X) \text{ch}(V) \right]_{(\dim X)}. \quad (1.10)$$

One of the remarkable things about the local index theorem is the existence of the limit on the left hand side of Eq. 1.10: a priori one would expect $\text{Tr } p_t(x, x)$ to diverge as $t^{-\dim X/2}$ as $t \rightarrow 0$.

1.2 \mathbb{Z}_2 -graded vector bundles and superconnections

Quillen[31] introduced superconnections to mathematics in 1980's as the appropriate generalisation of the notion of a connection for \mathbb{Z}_2 -graded vector. He introduced them motivated by K-theoretic considerations.

Let $V \rightarrow X$ be a smooth \mathbb{Z}_2 -graded vector bundle over a smooth manifold X . A superconnection ∇ on V is an odd derivation on $\Omega(X; V) \equiv \Omega(X) \otimes_{\Omega^0(X)} V$. The principal difference between superconnections and ordinary connections is that superconnections have terms in all cohomological degrees, whereas ordinary connections have only got terms in degree one. A superconnection may always be decomposed as

$$\nabla = \nabla + \sum_{i \neq 1} \omega_i, \quad (1.11)$$

where $\omega_i \in \Omega^i(X; \text{End}(V))$. There is a natural \mathbb{R}^\times -action on superconnections,

given by

$$\nabla^t = \nabla + \sum_{i \neq 1} |t|^{(1-i)/2} \omega_i. \quad (1.12)$$

The degree zero term is an odd endomorphism of V , and may be decomposed as

$$\omega_0 = \begin{pmatrix} & D^* \\ D & \end{pmatrix}.$$

The pair (V, ∇) represents the class in K-theory given by the sequence

$$D : V^0 \rightarrow V^1.$$

One may associate a curvature to superconnections in much the same way one does with ordinary connections: the curvature of a superconnection is computed by

$$\text{Curv}(\nabla) = \nabla^2,$$

and is purely algebraic. As with connections, one has a canonical de Rham representative of the Chern character of the K-theory class represented by (V, ∇) , computed by

$$\text{ch } \nabla = \text{Tr } e^{-\nabla^2}.$$

The differential form $\text{ch } \nabla$ is sharply peaked around the support of the K-theory class represented by (V, ∇) .

One may fashion Dirac operators out of superconnections in a manner entirely analogous to the construction of Dirac operators coupled to an ordinary connections: for a complex and hermitian \mathbb{Z}_2 -graded vector bundle

$V \rightarrow X$, the Dirac operator associated to a superconnection ∇ on V is given by

$$\mathcal{D}(\nabla) : \Gamma(\mathbb{S}(X) \otimes E) \xrightarrow{(\nabla^{\mathbb{S}} \otimes \mathbf{1}) \oplus (\mathbf{1} \oplus \nabla)} \Omega^{\bullet}(X; \mathbb{S}(X) \otimes E) \xrightarrow{c(\cdot)} \Gamma(\mathbb{S}(X) \otimes E).$$

The object of this dissertation is to study the index theory of Dirac operators coupled to superconnections.

1.3 A local index theorem for Dirac operators coupled to superconnections

Theorem 1.13. *Let X be a compact, spin and Riemannian manifold. Let $E \rightarrow X$ be a complex, hermitian and finite dimensional \mathbb{Z}_2 -graded vector bundle, with ∇ a unitary superconnection on E . Let $p_{t,s}(x, y)$ be the heat kernel associated to $\mathcal{D}(\nabla^s)$. Then*

$$\lim_{t \rightarrow 0} \text{Tr } p_{t,1/t}(x, x) \, dx = (2\pi i)^{-n/2} \left[\hat{A}(\Omega^X) \text{ch } \nabla \right]_{(n)},$$

where $n = \dim X$.

Ezra Getzler first proved this theorem in [22] using the Feynman-Kac formula. Notice the importance of the \mathbb{R}^{\times} -action (Eq. 1.12) in the statement: the limit on the left hand side of the local index theorem would not converge without it. We will see that the appearance of this \mathbb{R}^{\times} -action introduces some complications in generalising index theory to Dirac operators coupled to superconnections.

We provide a new proof of the local index theorem using heat equation methods and a scaling argument sketched in [17]. In order to explain the proof, we need to briefly review heat equation proofs in general. Minakshisundaram and Pleijel [28] were the first to study heat equations on manifolds. It was Patodi [29] who first applied heat equation techniques to proving the index theorem in various special cases. Gilkey [23] extended his ideas to prove the local index theorem in general. Atiyah, Bott and Patodi [1] provide a beautiful account of these proofs.

It is the McKean-Singer formula (Eq. 1.8) that lies at the heart of heat-equation proofs: it relates the the index to the asymptotics of the heat kernel, and thus provides a powerful means of attacking the index theorem. The heat kernel has a small t asymptotic expansion:

$$p(t; x, x) \sim \sum_{k \geq -\dim X/2} a_k t^k. \quad (1.14)$$

By the McKean-Singer formula,

$$\int_X a_0 dx = \text{index } \not{D}(\nabla).$$

One may thus prove the index theorem by showing that

$$\text{Tr } a_0 = (2\pi i)^{-\dim X/2} [\hat{A}(\Omega^X) \text{ch}(\nabla)]_{(\dim X)}.$$

At first sight, computing a_0 is a rather difficult proposition: it lies far into the asymptotic expansion of the heat kernel. Ezra Getzler [21] provided a breakthrough in heat equation proofs by finding an algebraic rescaling that

brings the degree zero term to the front of the asymptotic expansion, and thus making it easy to calculate.

Heat equation proofs of the index theorem thus reduce proving the index theorem to understanding the asymptotic expansion of the heat kernel associated with the Dirac operator. This is precisely where the difficulty lies in proving a local index theorem for Dirac operators coupled to superconnections. In the local index theorem (Th. 1.13) the \mathbb{R}^\times -action on superconnections (Eq. 1.12) is essential. We are thus faced with the task of understanding the small t asymptotic expansion of the heat kernel on the diagonal

$$p_{t,1/t}(x, x).$$

It is not clear at first that such an asymptotic expansion should even exist: the difficulty is in understanding the joint limit. In order to show the existence of a small t asymptotic expansion we use a scaling argument sketched in [17]. The idea is to use parabolic scaling to relate small time behaviour to the behaviour of the family of kernels under blowup. Understanding this behaviour turns out to be tractable: one may transfer the problem to a problem on \mathbb{R}^n , and use standard Fourier techniques to examine the behaviour of the family of heat kernels $p_{t,u}(x, y)$. The argument is given in Secs. 3.3.2, 3.4.

Having established the existence of a small time asymptotic expansion for $p_{t,1/t}(x, x)$, one may easily modify the Getzler scaling argument to prove the local index theorem.

1.4 Manifolds with boundary

It is natural to consider extending the index theorem to manifolds with boundary. Atiyah, Patodi and Singer investigated the problem in their 1975 series of papers [3, 4, 5], and they discovered a beautiful generalisation of the index theorem, where the boundary correction is given by a new *geometric* invariant – the η -invariant.

When extensions of the index theorem to manifolds with boundary were first considered, it was discovered that Dirac operators do not admit local boundary conditions. Atiyah, Patodi, and Singer realised that one could introduce global boundary conditions that allowed an interesting index problem to be constructed. Let Y be a smooth, Riemannian and spin odd dimensional manifold, with $\dim Y = n$. We define $\omega = i^{n(n+1)/2} \text{vol} \in \text{Cl}(Y)$, and note that it is an odd element of the Clifford algebra that squares to one. It commutes with the Dirac operator on Y , so that, with respect to the \mathbb{Z}_2 -grading on the spinors,

$$\omega \mathcal{D} = \begin{pmatrix} \omega \not{D} & \\ & \omega \not{D}^* \end{pmatrix}.$$

The diagonalisation of the laplacian diagonalises $\omega \mathcal{D}$. The spectrum of $\omega \not{D}$ is real and discrete, with finite dimensional eigenspaces. One may construct the operator $P : L^2(Y; S) \rightarrow L^2(Y; S)$ which projects L^2 -spinors to the span of the eigenspinors of $\omega \mathcal{D}$ with non-negative eigenvalues. Suppose now that Y is the boundary of some even-dimensional, compact and spin and Riemannian manifold X , and that there exists a collar neighbourhood U of the boundary diffeomorphic to $Y \times [0, 1)$, with the geometry on U pulled back from that

on $Y \times [0, 1)$. Atiyah-Patodi-Singer showed that the Dirac operator on X restricted to spinor fields $\psi \in L^2(X, S)$ such that $P\psi|_Y = 0$ is elliptic. Their theorem asserts

$$\int_X \hat{A}(\Omega^X) + \xi_Y = \text{index } \not{D}_X,$$

where \not{D}_X is the Dirac operator on X restricted to spinor fields ψ which satisfy the APS boundary conditions $P\psi|_Y = 0$, and ξ_Y is a new geometric invariant: the reduced η -invariant.

The η -invariant has a suggestive heuristic description: it may be interpreted as computing the discrepancy between the number of positive and negative eigenvalues of the Dirac operator. The reduced η -invariant ξ_Y is one of two invariants derived from the η -invariant. It is given by the formula

$$\xi_Y = \frac{\eta_Y + \dim \ker \not{D}_Y}{2}.$$

It is very often convenient to reduce the ξ -invariant modulo the integers. The resultant invariant is the τ -invariant:

$$\tau_Y = e^{2\pi i \xi_Y}.$$

It is natural to look for generalisations of the η -invariant to include Dirac operators coupled to superconnections, and show that these satisfy an APS-like theorem. A fundamental difficulty encountered is the necessity for considering the entire family $\mathcal{D}(\nabla^s)$ of Dirac operators. These do not necessarily commute, and any methods using a spectral decomposition become

difficult to apply.⁴

We are able to avoid this problem by using the APS theorem as a starting point. In the end we only define a τ -invariant, being unable to resolve an essential integer ambiguity

The basic idea that underlies the construction of the τ -invariant is that any superconnection ∇ may be homotoped to an ordinary connection ∇ . To any pair of superconnections on a \mathbb{Z}_2 -graded bundle, ∇_0, ∇_1 , one may associate a Chern-Simons like form

$$\alpha(\nabla_0, \nabla_1) = \int_0^1 dt \operatorname{Tr} \left((\nabla_0 - \nabla_1) e^{-((1-t)\nabla_0 + t\nabla_1)^2} \right).$$

The τ -invariant is most conveniently defined in terms of this form.

Definition 1.15. Let Y be an odd-dimensional, compact and spin Riemannian manifold, and $E \rightarrow Y$ be a complex and hermitian, \mathbb{Z}_2 -graded vector bundle with unitary superconnection ∇ . Write $\nabla = \nabla + \omega$, where ∇ is an ordinary connection on E , and $\omega \in \Omega(Y; \operatorname{End}(E))^{\operatorname{odd}}$. The τ -invariant associated with this data is

$$\tau_Y(\nabla) = \exp 2\pi i \left[\xi(\nabla) + (2\pi i)^{-n/2} \int_Y \hat{A}(\Omega^Y) \alpha(\nabla, \nabla) \right],$$

⁴Bismut and Cheeger [12] investigate η -invariants of Dirac operators coupled to superconnections. They use the standard definition of η -invariants in terms of the spectrum of the Dirac operator to define this, and investigate the behaviour of the resultant invariants under certain limits. The answers they obtain are all given in terms of certain terms of the asymptotic expansion of heat kernels associated to these Dirac operators. We do not use their definitions, as our principal concern is to have an η -invariant that obeys the appropriate APS theorem.

with $\xi(\nabla)$ the reduced η -invariant on Y , $n = \dim Y$, and $\alpha(\nabla, \nabla)$ the form defined above.

This is easily seen to be independent of choices, and almost by definition satisfies an APS-like theorem:

Theorem 1.16. *Suppose X is an even-dimensional compact and spin Riemannian manifold with boundary Y . Let $E \rightarrow X$ be a complex, hermitian, \mathbb{Z}_2 -graded vector bundle with unitary superconnection ∇ . Suppose further that there exists a collar neighbourhood $U \cong Y \times [0, 1)$ where the geometry is pulled back from that on Y . Then*

$$\tau_Y(\nabla) = \exp \left[(2\pi i)^{1-n/2} \int_X \hat{A}(\Omega^X) \text{ch}(\nabla) \right],$$

where $n = \dim X$.

The technique of using a homotopy to relate problems with superconnections to the associated problem with a connection is repeatedly exploited in studying the index theory of Dirac operators coupled to superconnections.

1.5 Families

We consider families of Dirac operators coupled to superconnections. In our context the useful notion of “family” is that of a Riemannian map. This is a smooth submersion $\pi : X \rightarrow Y$, along with a choice of metric $g^{X/Y}$ on the vertical tangent bundle $T(X/Y)$ and a projection $P : T(X) \rightarrow T(X/Y)$.

Such a map has a natural \mathbb{R}^\times -action: the Riemannian map $\pi^t : X \rightarrow Y$ is simply the original map π and projection P , but has vertical metric $|t|^{-1}g^{X/Y}$.

Families of Dirac operators were first studied in [9], where Atiyah and Singer show that the index of a family may be interpreted as an element of the K-theory of the base. Bismut [11] refined this theorem to a geometric construction: given a Riemannian family $\pi : X \rightarrow Y$, with compact and spin fibres, and a complex hermitian vector bundle $V \rightarrow X$ with unitary connection ∇ he constructs an infinite dimensional \mathbb{Z}_2 -graded vector bundle $\pi_! V \rightarrow Y$ with unitary superconnection $\pi_! \nabla$ that may be thought to represent the K-theory class of the index of the family of Dirac operators associated to the data. The degree zero part of the superconnection $\pi_! \nabla$ is the fibre Dirac operator. The Chern character of the family is computed by:

$$\lim_{t \rightarrow 0} \text{ch } \pi_!^t \nabla = (2\pi i)^{-n/2} \pi_* \left[\hat{A}(\Omega^\pi) \text{ch } \nabla \right],$$

where $n = \dim X/Y$. In particular, when $Y = \{\text{pt}\}$, the original index theorem is recovered.

We extend Bismut's construction to families of superconnections. Suppose now that we have a Riemannian map $\pi : X \rightarrow Y$, with the fibres compact and spin, and of dimension n , and a complex, hermitian, \mathbb{Z}_2 -graded vector bundle $E \rightarrow X$, with unitary superconnection ∇ . We define the pushforward of this bundle with its superconnection to the base, $\pi_! E \rightarrow Y$. This is an infinite dimensional, complex, hermitian \mathbb{Z}_2 -graded vector bundle, with fibre at some $y \in Y$ given by $\Gamma(\mathbb{S}_y(X/Y) \otimes E)$, where $\mathbb{S}(X/Y)$ is the bundle of vertical

spinors. The pushed-forward superconnection, $\pi_! \nabla$, is an extension of the Bismut superconnection, and has the fibrewise Dirac operator as its degree zero term. We prove a families index theorem for superconnections:

Theorem 1.17. *Let $X \rightarrow Y$ be a Riemannian fibration, with n -dimensional compact and spin fibres. Let $E \rightarrow X$ be a complex and hermitian \mathbb{Z}_2 -graded vector bundle with unitary superconnection ∇ . There exists a pushforward $(\pi_! E, \pi_! \nabla) \rightarrow Y$: this is a complex hermitian \mathbb{Z}_2 -graded vector bundle $\pi_! E \rightarrow Y$ with unitary superconnection $\pi_! \nabla$. The construction is functorial, in the sense that it commutes with pullback. Furthermore,*

$$\lim_{t \rightarrow 0} \text{ch } \pi_!^t \nabla = (2\pi i)^{-n/2} \pi_* \left[\hat{A}(\Omega^\pi) \text{ch } \nabla \right].$$

Here Ω^π denotes the curvature of the vertical Levi-Civita connection.

It should be noted that the \mathbb{R}^\times -action on the Riemannian map implicitly involves the action on ∇ defined before. In particular, restricting our attention to the degree zero part,

$$[\pi_!^t \nabla]_{(0)} = |t|^{1/2} \mathcal{D}^\pi(\nabla^{1/t}),$$

where, on the right hand side, $\mathcal{D}^\pi(\nabla)$ is the Dirac operator on the fibre. A corollary of the theorem is the following useful transgression formula:

Corollary 1.18. *With everything as above, one has*

$$\text{ch } \pi_!^t \nabla = (2\pi i)^{-n/2} \pi_* \left[\hat{A}(\Omega^\pi) \text{ch}(\nabla) \right] - d \int_0^t ds \text{Tr} \left(\frac{d\pi_!^t \nabla}{dt} e^{-(\pi_!^t \nabla)^2} \right) \Big|_s.$$

1.6 Determinant line bundles

We now turn our attention to determinant line bundles. Let $\pi : X \rightarrow Y$ be a Riemannian map with compact and spin fibres, $E \rightarrow X$ a complex and hermitian \mathbb{Z}_2 -graded vector bundle, with ∇ a unitary superconnection. To this family we wish to associate a line bundle $\det(\pi_! \nabla) \rightarrow Y$ with metric, section, and unitary connection, with the section representing the determinant of the Dirac operators coupled to the superconnections on the fibres. The connection of the bundle should be such that its curvature is the two-form part of $(2\pi i)^{-n/2} \pi_* \left[\hat{A}(\Omega^\pi) \text{ch}(\nabla) \right]$, where $n = \dim X/Y$. This construction should be functorial.

Quillen [30] constructed a geometric determinant line bundle for the $\bar{\partial}$ operator on a family of Riemann surfaces. To such a family $X \rightarrow Y$ he associated a line bundle $\det(\bar{\partial}) \rightarrow Y$ with section, metric, and holomorphic connection. Bismut and Freed [13, 14] extended his construction to arbitrary families of first order elliptic operators, assigning to such a family a line bundle with metric, section, and connection. For families of Dirac operators, they compute the curvature from Bismut's families index theorem and holonomy as the adiabatic limit of the η -invariant. Dai and Freed [15] give a different proof of the holonomy formula for determinant line bundles, interpreting the exponentiated η -invariant of an odd dimensional manifold with boundary as lying in the determinant line of the Dirac operator on the boundary.

The construction of Bismut and Freed carries through directly to define a geometric line bundle $\det \pi_! \nabla \rightarrow Y$ associated to a family of Dirac operators

coupled to superconnections. This line bundle has a canonical section, and metric. The construction of the required connection on the determinant bundle is new. We modify the connection obtained from the construction in [13] in a canonical fashion to give a connection on this bundle with curvature computed by the two-form part of the families index theorem. We compute the holonomy of this bundle in terms of the adiabatic limit of the exponentiated η -invariant.

As with η -invariants, the essential difficulty encountered in defining the determinant line bundle is the necessity of taking into account the entire family $\mathcal{D}^\pi(\nabla^t)$ of fibre Dirac operators occurring in the families index theorem. We are able to define the connection using the transgression formula (Eq. 1.18).

The holonomy of the determinant line bundle is calculated by the following theorem.

Theorem 1.19. *Let $X \rightarrow Y$ be a Riemannian fibration, with compact and spin fibres. Let $E \rightarrow X$ be a complex and hermitian \mathbb{Z}_2 -graded vector bundle, with unitary superconnection ∇ . There exists geometric line bundle associated to this data, $(\det(\pi^*\nabla), \nabla^{\det(\pi^*\nabla)}) \rightarrow Y$, with canonical section $\det \mathcal{D}^\pi(\nabla)$, which we call the determinant line bundle. Its construction commutes with pullbacks. Its curvature is computed by*

$$\text{Curv}(\nabla^{\det(\pi^*\nabla)}) = (2\pi i)^{-n/2} \pi_* \left[\hat{A}(\Omega^\pi) \text{ch}(\nabla) \right].$$

Furthermore, for a smooth based loop $\gamma : [0, 1] \rightarrow Y$, $\gamma(0) = \gamma(1)$:

$$\text{Hol}_\gamma(\nabla^{\det(\pi^*\nabla)}) = \text{a-lim } \tau_{\pi^{-1}\gamma}(\nabla),$$

where the right hand side is denotes the adiabatic limit of the τ -invariant of ∇ on $\pi^{-1}\gamma$.

1.7 Superconnections, determinants, and physics

In standard perturbative quantum field theory, the path integral over fermionic fields gives rise to the determinant of Dirac operators coupled to ordinary connections. The computation implicitly involves a trivialisation of the determinant line bundle of these operators. There are two obstructions to doing so, which in physics are termed the local and the global anomalies. Mathematically, these are the curvature and holonomy of the determinant line bundle.

The first to investigate the anomaly in terms of determinant line bundles were Atiyah and Singer [6]. They investigate anomalies in gauge theory, where one has an even dimensional space time X with a vector bundle $E \rightarrow X$ with connection parameterised by some space Y . Dirac operators are coupled to these connections, giving rise to a family of Dirac operators over Y . They compute the first Chern class of the determinant line bundle $\det D \rightarrow Y$ using the cohomological form of the families index theorem and produce an explicit two-form on Y representing this class. The anomaly is interpreted as relating to the variation of the determinant under infinitesimal gauge transformations.

Witten was the first to investigate torsion phenomena in anomalies. In [32] he investigates Dirac operators coupled to an $SU(2)$ connection in four dimensions. In this case the Dirac operators form a real family. The determinant

line bundle is a real bundle, and is characterised by a class in $H^1(\det \mathcal{D}, \mathbb{Z}_2)$. He calculates this class using the mod two index theorem, and interpreted his result in terms of the change of $\log \det \mathcal{D}$ under gauge transformations not connected to the identity. In [33] Witten turns his attention to gravitational anomalies. He examines the theory on S^{10} . In this case, the determinant of Dirac is complex, and can change phase when S^{10} is acted on by a diffeomorphism not connected to the identity. He constructs an eleven-dimensional manifold by building a circle bundle twisted by this change of phase, and finds that the global anomaly is related to the η -invariant of this manifold. He interprets this anomaly as the torsion in the first Chern class of the determinant line bundle with integer coefficients. Witten's work inspired a great deal of work on determinant line bundles [CHEEGER+SINGER]), and in particular Bismut and Freed's investigation of determinant line bundles [13, 14], where they construct canonical geometric determinant line bundles associated to families of Dirac operators, and derive a formula for the curvature in terms of the families index theorem, and the holonomy with an adiabatic limit of an η -invariant. In [19], Freed applies the results in [13, 14] to interpret anomalies and torsion in String Theory.

The action in eleven dimensional supergravity (and other supergravity theories) involves not just one-form couplings with fermions, but more complicated couplings involving higher degree forms. These may be interpreted as arising from Dirac operators coupled to superconnections, and one would expect the index theory these operators to come into play when investigating

anomalies in this theory. Freed and Moore [20] investigate this theory ignoring these more complicated terms, and show that the anomaly vanishes. In subsequent work Moore and Lukic [26] include these terms and show that the action may be interpreted in terms of Dirac operators coupled to superconnections. In order to compute the anomaly, they use the construction in [13], and the determinant line bundle thus has its curvature computed by the constant term in the small t asymptotic expansion of $\text{Tr} \exp(-t\mathcal{D}(\nabla)^2)$. They compute this perturbatively with careful analysis of certain path integrals.

The difficulties they encounter in computing the terms comes precisely because they do not encounter the \mathbb{R}^\times -action on superconnections that gives rise to the families index theorem. One might hope to re-interpret their work in a way to include this action, and thus easily calculate the anomaly in eleven dimensional supergravity.

1.8 Differential K-theory and superconnections

A \mathbb{Z}_2 -graded, complex, hermitian vector bundle with unitary superconnection (V, ∇) represents a class in K-theory, given by the sequence

$$V^0 \xrightarrow{[\nabla]_{(0)}} V^1,$$

where the map is given by the degree zero part of the superconnection. A natural application of the work in this thesis is to build a model for differential K-theory using superconnections.

Kevin Klonoff defines a model for differential K-theory for compact,

spin, Riemannian manifolds where the basic objects associated with a manifold X are finite dimensional \mathbb{Z}_2 -graded vector bundles over X with unitary (ordinary) connection and an odd degree form (V, ∇, ω) in his PhD thesis [25]. He shows that his model is equivalent to Hopkins and Singer's model of differential K-theory [24], and computes the push-forward to a point. One would expect to be able to extend this model so that the basic objects are \mathbb{Z}_2 -graded, complex, hermitian vector bundles with an arbitrary unitary superconnection, along with an odd differential form, and compute the pushforward to a point. The work in this manuscript should provide the basic theorems required. One may hope to extend the model to include pushforward in families: consider a Riemannian map $X \rightarrow Y$ with compact, Riemannian, spin fibres, and Y compact, Riemannian and spin. Given an element in the differential K-theory of the family represented by a \mathbb{Z}_2 -graded, complex, hermitian vector bundle V and unitary superconnection ∇ , one would like to compute its pushforward to Y in differential K-theory. Intuitively one would expect that for even dimensional fibres, this should be given by the pushforward bundle $\pi_! \nabla$ and superconnection $\pi_! \nabla$.

1.9 Organisation

The dissertation is organised as follows:

- Ch. 2 reviews the theory of \mathbb{Z}_2 -graded algebra and superconnections.
- Ch. 3 develops the necessary elliptic and parabolic theory of Dirac op-

erators coupled to superconnections. Our general approach follows [17], and the basic theory developed there carries through in a straightforward manner. In Sec. 3.3 we introduce the \mathbb{R}^\times -action on superconnections, and Secs. 3.3.2 and 3.4 use an argument inspired by [17] to show the existence of a small t asymptotic expansion for scaled heat kernels.

- Ch. 4 states and proves the local index theorem for such operators, using an approach very similar to [17],
- Ch. 5 considers manifolds with boundary, defining the τ -invariant for Dirac operators coupled to superconnections, and proving an APS theorem,
- Ch. 6 extends the theory to families, and proves a families index theorem. The proof for families is very similar to that for a single Dirac operator coupled to a superconnection.
- Ch. 7 discusses determinant line bundles associated to families of Dirac operators coupled to superconnections. We construct a geometric determinant line bundle with section and metric following the techniques in [13, 14]. New techniques are required to construct an appropriate connection. We compute the holonomy of the determinant line bundle by homotoping the line bundle back to a determinant line bundle for a family of Dirac operators coupled to an ordinary connection, and applying the theorem in [15].

Chapter 2

\mathbb{Z}_2 -graded vector bundles and superconnections

2.1 \mathbb{Z}_2 -graded algebras

A \mathbb{Z}_2 -graded vector space over a field k is a k -vector space together with a direct sum decomposition labelled by \mathbb{Z}_2 : the elements of degree zero are called even, and those of degree one, odd. Ungraded vector spaces may be thought of as \mathbb{Z}_2 -graded vector spaces that are purely even. It is convenient to denote the degree of a homogeneous element v of a \mathbb{Z}_2 -graded vector space by $|v|$. \mathbb{Z}_2 -graded vector spaces possess a grading endomorphism, ε , respecting the decomposition: it acts as the identity on even elements, and as the negative of the identity on the odd elements. A \mathbb{Z}_2 -graded algebra is an algebra that is also a \mathbb{Z}_2 -graded vector space, with a product that respects the grading in the sense that the grading endomorphism is a homomorphism, in other words $\varepsilon(v \cdot w) = \varepsilon(v)\varepsilon(w)$ for homogeneous elements v, w of a \mathbb{Z}_2 -graded algebra.

Linear algebra may be extended in a systematic way to the grading of \mathbb{Z}_2 -graded vector spaces and algebras. Deligne and Morgan provide an excellent exposition in [16]. We will not be interested in developing the general theory here, but will simply describe those constructions that will be necessary

for us.

Let A be a \mathbb{Z}_2 -graded algebra. The commutator on A , denoted by $[\cdot, \cdot] : A \times A \rightarrow A$, is defined by

$$[a, b] = a \cdot b - (-1)^{|a||b|} b \cdot a,$$

on homogeneous elements $a, b \in A$. As usual, pairs of elements that the commutator vanishes on are said to commute. A trace on a \mathbb{Z}_2 -graded algebra is a linear map $\text{Tr} : A \rightarrow k$ that vanishes on commutators.

The tensor product of two \mathbb{Z}_2 -graded vector spaces is simply their tensor product as vector spaces, graded by the tensor product of their grading endomorphisms. For any pair of \mathbb{Z}_2 -graded vector spaces V and W , one has the canonical isomorphism $V \otimes W \xrightarrow{\sim} W \otimes V$ given by

$$v \otimes w \mapsto (-1)^{|v||w|} w \otimes v,$$

where $v \in V, w \in W$. If A, A' are two \mathbb{Z}_2 -graded algebras, then the product on the tensor product algebra $A \otimes A'$ is given by

$$(a \otimes a') \cdot (b \otimes b') = (-1)^{|a'||b|} (a \cdot b) \otimes (a' \cdot b')$$

on homogeneous elements $a, b \in A, a', b' \in A'$.

Example 2.1 (The endomorphism algebra). The basic example of a \mathbb{Z}_2 -graded algebra is the algebra $\text{End}(V)$ of endomorphisms of a \mathbb{Z}_2 -graded vector space V . The grading is induced by identifying $\text{End}(V) \cong V \otimes V^*$: the even elements

of $\text{End}(V)$ are those that preserve the grading of V , and the odd elements those that reverse it. One defines the trace map by the sequence

$$\text{Tr} : \text{End}(V) \xrightarrow{\sim} V \otimes V^* \xrightarrow{\sim} V^* \otimes V \rightarrow k,$$

where the last map is the pairing between V and its dual. In terms of the ungraded trace on the endomorphism algebra, Tr_0 , one has

$$\text{Tr}(X) = \text{Tr}_0(\varepsilon(X)).$$

Example 2.2 (The tensor algebra). Let V be an ungraded vector space. The tensor algebra of V ,

$$\bigotimes V = \bigoplus_{k \geq 0} V^{\otimes k},$$

is a \mathbb{Z}_2 -graded algebra, the grading being given by degree mod 2, and the multiplication being tensor multiplication.

Example 2.3 (The exterior algebra). Let V be an ungraded vector space of dimension n . Then $\bigwedge V$ is a *commutative* \mathbb{Z}_2 -graded algebra of dimension 2^n , where the grading is given by the degree mod 2, and the multiplication is the wedge product.

Example 2.4 (Clifford algebras). Let (V, g) be a real¹ vector space of dimension n , with a non-degenerate symmetric bilinear form $g : V \otimes V \rightarrow \mathbb{R}$ of signature (p, q) . The Clifford algebra $\text{Cl}(V, g)$ is defined by

$$\bigotimes V / (u \otimes v + v \otimes u + 2g(u, v)),$$

¹Clifford algebras may be defined over any field, although care must be taken in characteristic two. We will only be interested in Clifford algebras over \mathbb{R} and their complexification, and we thus restrict our attention to these.

where the u, v run over all elements of V . This is a \mathbb{Z}_2 -graded algebra, with the grading and multiplication induced from $\bigotimes V$. It is not hard to see that $\dim \text{Cl}(V, g) = 2^n$. Indeed, suppose $\{v_i\}_{i=1\dots n}$ is a basis for v . Then $\{v_{i_1} v_{i_2} \cdots v_{i_k}, i_1 < \dots < i_k, 0 \leq k \leq n\}$ gives a basis for $\text{Cl}(V, g)$. The Clifford algebra is canonically isomorphic to the exterior algebra as a \mathbb{Z}_2 -graded vector space (although not as an algebra), the isomorphism sending

$$v_{i_1} v_{i_2} \cdots v_{i_k} \mapsto v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}.$$

It will be convenient to denote the image of $a \in \text{Cl}(V)$ in $\bigwedge V$ by \hat{a} . The exterior algebra of V is a $\text{Cl}(V, g)$ module, with the action defined by

$$c(v) = e(v) - \iota(v),$$

where $v \in V$, e is exterior multiplication, and ι is interior multiplication with respect to g . The Clifford algebra is a deformation of the exterior algebra. Define the family of Clifford algebras $\text{Cl}^\varepsilon(V, g) \equiv \text{Cl}(V, \varepsilon^2 g)$, where $\varepsilon \in \mathbb{R}$: when $\varepsilon \neq 0$, $\text{Cl}^\varepsilon(V, g) \cong \text{Cl}(V, g)$ as \mathbb{Z}_2 -graded algebras, but $\text{Cl}^0(V, g) = \bigwedge V$. Denote the action of $\text{Cl}^\varepsilon(V, g)$ on $\bigwedge V$ by c_ε . The following easy lemma will be important.

Lemma 2.5. Let $\hat{U}_\varepsilon : \bigwedge V \rightarrow \bigwedge V$ be defined by

$$\hat{U}_\varepsilon : \hat{v} \mapsto \varepsilon^{-|\hat{v}|} \hat{v}, \tag{2.6}$$

on homogeneous elements. Then

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^k c_\varepsilon \left(\hat{U}_\varepsilon(\hat{a}) \right) = \begin{cases} 0 & \text{if } k > |\hat{a}|, \\ e(\hat{a}) & \text{if } k = |\hat{a}|, \\ \infty & \text{if } k < |\hat{a}|, \end{cases}$$

where $\hat{a} \in \bigwedge V$ is homogeneous, $|\hat{a}|$ is the degree (in \mathbb{Z}) of \hat{a} , and we have implicitly identified the \mathbb{Z}_2 -graded vector spaces $\text{Cl}^\varepsilon(V, g)$ and $\bigwedge V$.

Let us restrict our attention to a particular Clifford algebra $\text{Cl}(V, g)$. It is not hard to see that the degree two elements close up under the commutator to form a Lie algebra isomorphic to $\mathfrak{so}_n(V, g)$. Exponentiating these inside $\text{Cl}(V, g)$ we get the Lie group of all even unit length elements, which we shall call $\text{Spin}(V, g)$. It is a double cover of $SO(V, g)$.

We now proceed to discuss the structure of Clifford algebras and modules. Define $\text{Cl}^\mathbb{C}(V, g) \equiv \text{Cl}(V, g) \otimes_{\mathbb{R}} \mathbb{C}$. Define $\text{Cl}(n) \equiv \text{Cl}(\mathbb{R}^n)$, where \mathbb{R}^n is endowed with its usual inner product, and define $\text{Cl}^\mathbb{C}(n)$ similarly. We have the following structure theorem.

Proposition 2.7. [2] The complexified Clifford algebras $\text{Cl}^\mathbb{C}(k)$ have the following structure:

1. $\text{Cl}^\mathbb{C}(2n) \cong \text{End}(S)$, where S is a \mathbb{Z}_2 -graded vector space of dimension 2^n ,
2. $\text{Cl}^\mathbb{C}(2n+1) \cong \text{Cl}^\mathbb{C}(2n) \otimes \text{Cl}^\mathbb{C}(1)$.

Remark 2.8. The real Clifford algebras also have a structure theorem, where the isomorphism classes repeat with period eight.

Proposition 2.7 allows us to conclude that in every dimension, the complexified Clifford algebras have a unique irreducible representation up to isomorphism, which we call the spinors, and denote by S_n . In odd dimensions,

this is the basic irreducible representation of $\text{Spin}(2n+1)$, but in even dimension the spinors decompose as a \mathbb{Z}_2 -graded representation of $\text{Spin}(2n)$, with isomorphic even and odd components. In this case the grading endomorphism is given by the element $\omega_n = i^{n(n+1)/2} e_1 \dots e_n$, where $\{e_i\}$ is an orthonormal basis for \mathbb{R}^n . We declare the \mathbb{Z}_2 -graded spinors \mathbb{S}_n for $\text{Spin}(n)$ to be the module of spinors when n is even, and to be the \mathbb{Z}_2 -graded $\text{Cl}(n)$ module $\mathbb{S}_n = S_n \oplus S_n$, with the first factor declared even, and the second odd, when n is odd. Note that Clifford multiplication by odd degree elements acts via odd endomorphisms of \mathbb{S}_n . In particular, when n is odd, ω_n is an odd endomorphism that squares to one, so is an isomorphism from $\mathbb{S}_n^{\text{ev}} \rightarrow \mathbb{S}_n^{\text{odd}}$.

2.2 \mathbb{Z}_2 -graded vector bundles

It is natural to consider bundles of \mathbb{Z}_2 -graded vector spaces. Up to this point we have been working over any field k , but we shall now work exclusively over \mathbb{R} and \mathbb{C} . Let X be a smooth manifold. A \mathbb{Z}_2 -graded vector bundle over X is a vector bundle with a decomposition as a direct sum of vector bundles indexed by \mathbb{Z}_2 . For a given \mathbb{Z}_2 -graded vector bundle $V \rightarrow X$, the bundle $\text{End}(V) \rightarrow X$ is a \mathbb{Z}_2 -graded algebra bundle, and has a distinguished section given by the grading endomorphism $\varepsilon \in \Gamma(\text{End}(V))$ which is the grading endomorphism on each fibre. The constructions detailed in the previous section all carry through to families.

Example 2.9 (Differential forms). Every smooth manifold X has a canonical \mathbb{Z}_2 -graded differential algebra associated with it: the bundle of differential

forms $\Omega^\bullet(X)$ with differential d . These are graded by the cohomological degree reduced mod 2, and the multiplication is the wedge product. It is a commutative algebra. Given a \mathbb{Z}_2 -graded vector bundle $V \rightarrow X$, one may form the \mathbb{Z}_2 -graded vector bundle of forms with value in V :

$$\Omega(X; V)^\bullet = \Omega(X) \otimes_{\Omega^0(X)} V.$$

It is a left module for the \mathbb{Z}_2 -graded algebra

$$\Omega(X; \text{End}(V))^\bullet = \Omega(X) \otimes_{\Omega^0(X)} \text{End}(V).$$

Example 2.10 (Clifford bundles). The discussion of Clifford algebras and Clifford modules in Sec. 2.4 may readily be extended to families. For a compact, Riemannian manifold X , we may form the associated Clifford bundle $\text{Cl}(X)$, which is the smooth bundle of \mathbb{Z}_2 -graded algebras with fibre at a point $x \in X$ given by $\text{Cl}(T_x^*X, g)$, where g is the Riemannian metric on X . We say a \mathbb{Z}_2 -graded vector bundle $E \rightarrow X$ is a Clifford module when it has a smooth $\text{Cl}(X)$ action on the fibres. We insist that $\text{Cl}(X)$ acts in a \mathbb{Z}_2 -graded sense. A Riemannian and spin manifold X has, upon choice of a spin structure, a canonical Clifford module associated to its Clifford algebra: the bundle of \mathbb{Z}_2 -graded spinors $\mathbb{S}(X) \rightarrow X$. Fibrewise, these are just the spin modules discussed in 2.4. We denote the $\text{Cl}^\mathbb{C}(X)$ action on $\mathbb{S}(X)$ by $c(\cdot)$. Any \mathbb{Z}_2 -graded $\text{Cl}^\mathbb{C}(X)$ module $M \rightarrow X$, may be decomposed in a non-canonical way as $M \simeq \mathbb{S}(X) \otimes V$. The spinors carry a canonical hermitian structure and connection induced from the Riemannian structure on X and the Levi-Civita connection.

2.3 Superconnections

Quillen [31] generalised the notion of a connection to the context of \mathbb{Z}_2 -graded vector bundles. Let X be a smooth manifold and $V \rightarrow X$ be a smooth \mathbb{Z}_2 -graded vector bundle. A *superconnection* on V is an odd derivation on $\Omega(X; V)^\bullet$. The space of superconnections on V is affine, and modeled on $\Omega(X; \text{End}(V))^{\text{odd}}$: every superconnection ∇ may be decomposed as

$$\nabla = \nabla + \sum_i \omega_i, \quad (2.11)$$

where ∇ is an ordinary connection on V , and the $\omega_i \in \Omega^i(X; \text{End}(V)^\bullet)$ are odd. The *curvature* of a superconnection ∇ is defined by

$$\text{Curv}(\nabla) \equiv \nabla^2.$$

A simple computation shows $\text{Curv}(\nabla) \in \Omega(X; \text{End}(V))^{\text{ev}}$.

There is a notion of unitarity for superconnections on complex and hermitian \mathbb{Z}_2 -graded vector bundles. Let $V \rightarrow X$ is a complex and hermitian \mathbb{Z}_2 -graded vector bundle, and ∇ be a superconnection on V . Locally we may write

$$\nabla = d + \sum_{i,j} \omega_{ij} \otimes E_{ij}$$

where $\omega_{ij} \in \Omega^i(X)$, $E_{ij} \in \text{End}^j(V)$, and $i + j = 1 \pmod{2}$. Then ∇ is *unitary* if

$$E_{ij} \text{ is } \begin{cases} \text{hermitian} & \text{if } i = 0, 3 \pmod{4}, \\ \text{anti-hermitian} & \text{if } i = 1, 2 \pmod{4}. \end{cases}$$

This restricts to the usual notion of unitary if ∇ is a connection.

The space of superconnections admits a natural \mathbb{R}^\times -action. In terms of the decomposition 2.11 the action is given by

$$\nabla^s = \nabla + \sum_i |s|^{(1-i)/2} \omega_i, \quad (2.12)$$

where $s \in \mathbb{R}^\times$. This action combines with the \mathbb{R}^\times action on $\text{Cl}(X)$ (Eq. 2.6) to give the following useful lemma.

Lemma 2.13. *Writing $\nabla^u = \nabla + \omega$, one has*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon c_\varepsilon(\hat{U}_\varepsilon \omega_{s/\varepsilon^2}) = e(\omega_s).$$

Proof. This follows immediately from 2.5, and noting

$$\varepsilon \omega_{s/\varepsilon^2} = s^{1/2} \omega_0 + \varepsilon \omega_1 + \varepsilon^2 s^{-1/2} \omega_2 + \dots$$

□

2.3.1 The Chern character

Quillen [31] associated a Chern character form to a superconnection. Let $V \rightarrow X$ be a finite-dimensional, \mathbb{Z}_2 -graded vector bundle with superconnection ∇ over a smooth manifold X . The Chern character form² associated to (V, ∇) is

$$\text{ch } \nabla \equiv \text{Tr} \left(\exp -\nabla^2 \right).$$

²A factor of $2\pi i$ in the exponent is often used in the definition.

It is a closed differential form. Given a one parameter family of superconnections ∇_t , one has the transgression formula

$$\frac{d}{dt} \text{ch } \nabla_t = d \text{Tr} \left[e^{-(\nabla_t)^2} \frac{d}{dt} \nabla_t \right]. \quad (2.14)$$

This shows that the class in de Rham cohomology represented by $\text{ch } \nabla$ is independent of the choice of superconnection on V .

2.4 Dirac operators

Suppose now that X is Riemannian and spin. Let V be a \mathbb{Z}_2 -graded vector bundle, with superconnection ∇ . Then the *Dirac operator* associated to ∇ is defined by the sequence

$$\mathcal{D}(\nabla) : \Gamma(\mathbb{S}(X) \otimes E) \xrightarrow{(\nabla^{\mathbb{S}} \otimes \mathbf{1}) \oplus (\mathbf{1} \oplus \nabla)} \Omega^\bullet(X; \mathbb{S}(X) \otimes E) \xrightarrow{c(\cdot)} \Gamma(\mathbb{S}(X) \otimes E).$$

If V is complex and hermitian, and ∇ is unitary, then $\mathcal{D}(\nabla)$ is formally self-adjoint with respect to the L^2 pairing on $L^2(X; V)$.

Chapter 3

Elliptic theory for Dirac operators coupled to superconnections

This chapter develops the elliptic theory for Dirac operators coupled to superconnections. To a large extent, we follow the treatment in [17]. While the extension to superconnections is mostly straightforward, it is critical to pay close attention to the dependence of the estimates on the parameter in the \mathbb{R}^\times -action on superconnections given in Eq. 2.12. The nature of the dependence is crucial in the proof of the existence of a small t asymptotic expansion of the heat kernel associated with a superconnection, when the dependence on time is coupled to this action.

3.1 Introduction

Let X be a smooth, compact and spin Riemannian manifold. Let $E \rightarrow M$ be a complex, hermitian and finite dimensional \mathbb{Z}_2 -graded vector bundle with a unitary superconnection ∇ . The associated Dirac operator is formally self-adjoint:

$$\langle \mathcal{D}(\nabla)\phi, \psi \rangle_{L^2} = \langle \phi, \mathcal{D}(\nabla)\psi \rangle_{L^2},$$

where $\phi, \psi \in \Gamma(\mathbb{S} \otimes E)$, and $\langle \cdot, \cdot \rangle_{L^2}$ denotes the L^2 pairing of sections of $\mathbb{S}(X) \otimes E$ induced by the the Riemannian metric on X and the hermitian structure on E .

Proposition 3.1 (Weitzenböck formula). *Writing $\nabla = \nabla + \omega$, where ∇ is a connection on $E \rightarrow X$, and $\omega \in \Omega(X; \text{End}(E))$, one has*

$$\begin{aligned} \mathcal{D}(\nabla)^2 &= \nabla^* \nabla + \frac{R}{4} + Q, \\ Q &= c(\Omega^E) + [\mathcal{D}(\nabla), c(\omega)] + c(\omega)^2, \end{aligned} \tag{3.2}$$

where Ω^E is the twisting curvature of ∇ , and $\nabla^* \nabla$ is the covariant laplacian constructed from $\nabla^{\mathbb{S}} \otimes 1 + 1 \otimes \nabla$.

Proof. Notice that

$$\mathcal{D}(\nabla) = \mathcal{D}(\nabla) + c(\omega).$$

Thus

$$\mathcal{D}(\nabla)^2 = \mathcal{D}(\nabla)^2 + [\mathcal{D}(\nabla), c(\omega)] + c(\omega)^2.$$

We need only see that

$$\mathcal{D}(\nabla)^2 = \nabla^* \nabla + \frac{R}{4} + c(\Omega^E).$$

This is the standard Weitzenböck formula, so we are done. \square

Remark 3.3. The appearance of the commutator term $[\mathcal{D}(\nabla), \omega]$ in Eq. 3.2 shows that laplacians built from superconnections are different from those built from ordinary connections: in the latter case one may always choose

coordinates that the operator is purely a sum of a second order and zero'th order differential operator, whereas in our case one may in general not avoid the appearance of a first order term. This presents few difficulties in the analysis.

3.2 The spectral theory of Dirac operators

For the remainder of the chapter we will fix a decomposition $\nabla^s = \nabla + \omega_s$, where we recall the superscript denotes the \mathbb{R}^\times -action on superconnections (Eq. 2.12), and write $\mathbb{S} = \mathbb{S}(X)$. We abbreviate $\mathcal{D}(\nabla^s)$ as \mathcal{D}_s , and $\mathcal{D}(\nabla)$ as \mathcal{D} . We write \mathcal{D} for \mathcal{D}_1 . Throughout the chapter we will take assume $s \in \mathbb{R}^\times$. For the most part we will be content to regard s as fixed, but for certain key estimates it will be important to note how the constants depend on s . It will also be convenient to define the family of positive sesquilinear forms

$$B_s[\psi, \phi] = \langle \mathcal{D}_s \psi, \mathcal{D}_s \phi \rangle_{L^2}$$

on the Sobolev space $H^1(X; \mathbb{S} \otimes E)$. We begin with a fundamental estimate:

Proposition 3.4. *Let $\psi \in H^1(X; \mathbb{S} \otimes E)$. Then there exist positive $\alpha_s, C_s \in \mathbb{R}[|s|^{1/2}, |s|^{-1/2}]$ such that*

$$\alpha_s \|\psi\|_{H^1}^2 \leq B[\psi, \psi] + C_s \langle \psi, \psi \rangle_{L^2}.$$

Proof. The Weitzenböck formula, Eq. 3.2 shows

$$\begin{aligned} B[\psi, \psi] &= \langle \nabla \psi, \nabla \psi \rangle_{L^2} + \frac{R}{4} \langle \psi, \psi \rangle_{L^2} + \langle c(\Omega^E) \psi, \psi \rangle_{L^2} \\ &\quad + \langle [\mathcal{D}, c(\omega_s)] \psi, \psi \rangle_{L^2} + \langle c(\omega_s)^2 \psi, \psi \rangle_{L^2}, \end{aligned}$$

for $\psi \in H^1(X; \mathbb{S} \otimes E)$. Adding $\langle \psi, \psi \rangle_{L^2}$ to both sides, one obtains

$$\begin{aligned} \|\psi\|_{H^1}^2 &= B[\psi, \psi] - \left(\frac{R}{4} - 1\right) \langle \psi, \psi \rangle_{L^2} - \langle c(\Omega^E)\psi, \psi \rangle_{L^2} \\ &\quad - \langle c(\omega_s)^2\psi, \psi \rangle_{L^2} - \langle [\mathcal{D}, c(\omega_s)]\psi, \psi \rangle_{L^2}. \end{aligned}$$

The compactness of X , and the fact that $c(\Omega^E)$ and $c(\omega_s)^2$ are both algebraic, and that ω_s is a form-valued rational function in $|s|^{1/2}$ allows one to find a non-negative $C'_s \in \mathbb{R}[|s|^{-1/2}, |s|^{1/2}]$ such that

$$C'_s \langle \psi, \psi \rangle_{L^2} \geq - \left(\frac{R}{4} - 1\right) \langle \psi, \psi \rangle_{L^2} - \langle c(\Omega^E)\psi, \psi \rangle_{L^2} - \langle c(\omega_s)^2\psi, \psi \rangle_{L^2}.$$

Similar considerations allow us to find a non-negative $C''_s \in \mathbb{R}[|s|^{-1/2}, |s|^{1/2}]$ such that

$$C''_s \operatorname{Re} \langle \nabla \psi, \psi \rangle_{L^2} \geq - \langle [\mathcal{D}, c(\omega_s)]\psi, \psi \rangle_{L^2},$$

so that

$$\|\psi\|_{H^1}^2 \leq B[\psi, \psi] + C'_s \|\psi\|_{L^2}^2 + C''_s \operatorname{Re} \langle \nabla \psi, \psi \rangle_{L^2}.$$

We now apply “Cauchy’s inequality with a β ”,

$$2\operatorname{Re} \langle \psi, \phi \rangle_{L^2} \leq \beta \langle \psi, \psi \rangle_{L^2} + \frac{1}{\beta} \langle \phi, \phi \rangle_{L^2},$$

to $\operatorname{Re} \langle \nabla \psi, \psi \rangle_{L^2}$, choosing $\beta < \frac{1}{2}$, to obtain

$$\alpha_s \|\psi\|_{H^1}^2 \leq B[\psi, \psi] + C_s \langle \psi, \psi \rangle_{L^2},$$

where $\alpha, C_s \in \mathbb{R}[|s|^{-1/2}, |s|^{1/2}]$ are positive. \square

Remark 3.5. Applying this to the special case where the superconnection is a connection, and being careful with the choice of the constants one obtains Gårding’s inequality.

The following corollary is an easy consequence of the proposition.

Corollary 3.6. *The family of inner products*

$$\langle\langle\phi, \psi\rangle\rangle_s = B_s[\phi, \psi] + C_s\langle\phi, \psi\rangle_{L^2}, \quad (3.7)$$

for $\phi, \psi \in H^1(X; \mathbb{S} \otimes E)$, is equivalent to the usual inner product on $H^1(X; \mathbb{S} \otimes E)$.

3.2.1 The spectrum of the laplacian

We now investigate the spectrum of the operator \mathcal{D}_s^2 , for fixed s .

Proposition 3.8. *The operator $T : H^1(X; \mathbb{S} \otimes E) \rightarrow H^{-1}(X; \mathbb{S} \otimes E)$ defined by $T = \mathcal{D}_s^2 + C_s$ is an isomorphism. Here C_s is as in proposition 3.4.*

Proof. The operator T is a second order differential operators, and thus a bounded linear operator from $H^1(X; \mathbb{S} \otimes E)$ to $H^{-1}(X; \mathbb{S} \otimes E)$. Prop. 3.4 shows that

$$\|T\phi\|_{H^{-1}}^2 \geq \alpha_s \|\phi\|_{H^1}^2,$$

for $\phi \in H^1(X; \mathbb{S} \otimes E)$. Consequently T is injective with closed range. We need to show T is onto.

Consider the functional e_f on $H^1(X; \mathbb{S} \otimes E)$ given by $e_f(\psi) = (f, \psi)$, where $f \in H^{-1}(X; \mathbb{S} \otimes E)$, and (\cdot, \cdot) denotes the pairing between a vector space and its dual. This is a bounded functional, so the Riesz representation theorem allows one to find a $\phi \in H^1(X; \mathbb{S} \otimes E)$ such that $e_f(\psi) = \langle\langle\phi, \psi\rangle\rangle_s$.

But then $e_f(\psi) = \langle T(\phi), \psi \rangle_{L^2}$ for any $\psi \in H^1(X, E)$, so that $T(\phi) = f$. We can do this for any $f \in H^{-1}(X, E)$, so T is onto.

Finally, Prop. 3.4 shows that T^{-1} is bounded. \square

We may now consider the operator $S : L^2(X; \mathbb{S} \otimes E) \rightarrow L^2(X; \mathbb{S} \otimes E)$ defined by the sequence

$$S : L^2(X; \mathbb{S} \otimes E) \longrightarrow H^{-1}(X; \mathbb{S} \otimes E) \xrightarrow{T^{-1}} H^1(X; \mathbb{S} \otimes E) \longrightarrow L^2(X; \mathbb{S} \otimes E), \quad (3.9)$$

where the first map is the inclusion of $L^2(X; \mathbb{S} \otimes E) \rightarrow H^{-1}(X; \mathbb{S} \otimes E)$, and the last the inclusion $H^1(X; \mathbb{S} \otimes E) \rightarrow L^2(X; \mathbb{S} \otimes E)$.

Proposition 3.10. *The operator $S : L^2(X; \mathbb{S} \otimes E) \rightarrow L^2(X; \mathbb{S} \otimes E)$ defined in Eq. 3.9 is positive, compact and self-adjoint.*

Proof. The positivity and self-adjointness are obvious from the fact that T shares these properties. The operator is compact as the two inclusions are compact. \square

We may now apply the spectral theorem for compact, self-adjoint operators to show the existence of a non-increasing countable sequence of real numbers $\xi_n \geq 0$, and an orthonormal basis $\phi_n \in L^2(X; \mathbb{S} \otimes E)$ of $L^2(X; \mathbb{S} \otimes E)$, such that:

1. $S(\phi) = \sum_i \xi_i \langle \phi_i, \phi \rangle_{L^2}$,
2. $\{\xi_n\} \in l^2(\mathbb{R})$,

$$3. \|S\|_{L^2}^2 = \sum_i \xi_i^2.$$

The following proposition is clear.

Proposition 3.11. *There exists a sequence $\{\psi_n\} \subset L^2(X; \mathbb{S} \otimes E)$ satisfying:*

1. *The ψ_n form an orthonormal basis for $L^2(X; \mathbb{S} \otimes E)$,*
2. *$\mathcal{D}^2 \psi_n = \lambda_n \psi_n$,*
3. *the sequence of λ_n is real and non-decreasing,*
4. *for any $\lambda > 0$, there are at most a finite number of $\lambda_n < \lambda$,*
5. *$\lambda_n \geq 0$ for all n .*

Our goal now is to show that the ψ_n are actually smooth.

Proposition 3.12. *If $\psi \in H^{k+1}(X; \mathbb{S} \otimes E)$,*

$$\alpha_s \|\psi\|_{H^{k+1}}^2 \leq \|\mathcal{D}_s \psi\|_{H^k}^2 + C_s \|\psi\|_{H^k}^2,$$

where α, C are non-negative polynomials in $|s|^{\pm 1/2}$.

Proof. The case $k = 0$ is done in Prop. 3.4. We now need to compute the ungraded commutator

$$[\mathcal{D}_s, \nabla]_0.$$

We first note that $[\nabla, \nabla]_0$ is purely algebraic (i.e. an endomorphism of $\mathbb{S} \otimes E$). This follows as in the proof that the curvature of a superconnection is

purely algebraic: one need merely note that tensor multiplication and Clifford multiplication commute.

We use induction to prove the proposition. Suppose we have

$$\alpha_s \|\psi\|_{H^{k+1}}^2 \leq \|\mathcal{D}_s \psi\|_{H^k}^2 + C_s \|\psi\|_{H^k}^2,$$

for all $\psi \in H^{k+1}(X, \mathbb{S} \otimes E)$. Let $\psi \in H^{k+2}(X, \mathbb{S} \otimes E)$ so that $\nabla \psi \in H^{k+1}(X, \mathbb{S} \otimes E)$. We may apply the inequality to show

$$\alpha_s \|\nabla \psi\|_{H^{k+1}}^2 \leq \|\mathcal{D}_s \nabla \psi\|_{H^k}^2 + C_s \|\nabla \psi\|_{H^k}^2.$$

Using the fact that $[\mathcal{D}, \nabla]_0$ is algebraic, we see that

$$\|\mathcal{D}_s \nabla \psi\|_{H^k}^2 \leq C'_s \|\nabla \mathcal{D}_s \psi\|_{H^k}^2.$$

We thus obtain, for some choice of positive $\alpha_s, C_s \in \mathbb{R}[|s|^{-1/2}, |s|^{1/2}]$,

$$\alpha_s \|\nabla \psi\|_{H^{k+1}}^2 \leq \|\nabla \mathcal{D}_s \psi\|_{H^k}^2 + C \|\nabla \psi\|_{H^k}^2.$$

Combining this with the estimate for $k = 0$,

$$\alpha_s \|\psi\|_{H^1}^2 \leq \|\mathcal{D}_s \psi\|_{L^2}^2 + C_s \|\psi\|_{L^2}^2,$$

we obtain the result:

$$\alpha_s \|\psi\|_{H^{k+2}}^2 \leq \|\mathcal{D}_s \psi\|_{H^{k+1}}^2 + C_s \|\psi\|_{H^{k+1}}^2.$$

□

One may strengthen the Prop. 3.12 to the following proposition using a difference quotients argument ([17], Chapter 3).

Proposition 3.13. *Suppose $\psi \in H^k(X; \mathbb{S} \otimes E)$, $\mathcal{D}(\nabla) \in H^k(X; \mathbb{S} \otimes E)$. Then $\psi \in H^{k+1}(X; \mathbb{S} \otimes E)$, and*

$$\alpha_s \|\psi\|_{H^{k+1}}^2 \leq \|\mathcal{D}_s \psi\|_{H^k}^2 + C_s \|\psi\|_{H^k}^2,$$

where α, C are non-negative rational functions in $|s|^{1/2}$.

Prop 3.13 along with the Sobolev lemma now shows that all the eigenfunctions ψ_n are smooth.

3.3 Heat operators and their kernels

We will now apply the results of the previous section to study the equation

$$\partial_t \phi(t, x) + \mathcal{D}_s^2 \phi(t, x) = 0, \quad (3.14)$$

where ϕ is a time varying section of $\mathbb{S}(X) \otimes E$. We call Eq. 3.14 the heat equation associated to ∇_s . For a given initial condition $\phi_0 \in \Gamma(\mathbb{S} \otimes E)$, Eq. 3.14 is formally solved by

$$e^{-t\mathcal{D}_s^2} \phi_0 \equiv \sum_n e^{-t\lambda_n} \langle \phi, \psi_n \rangle_{L^2} \psi_n. \quad (3.15)$$

The operator $e^{-t\mathcal{D}_s^2}$ is called the heat semigroup associated to ∇_s . Our first task is to investigate the regularity of solutions to Eq. 3.14.

Proposition 3.16. *There exists a unique solution to*

$$\begin{aligned} \partial_t \phi(t, x) + \mathcal{D}_s^2 \phi(t, x) &= 0, \\ \phi(0, x) &= \phi_0, \end{aligned} \quad (3.17)$$

where $\phi_0 \in L^2(X; \mathbb{S} \otimes E)$ given by

$$\phi(t, x) = \exp[-t\mathcal{D}_s^2]\phi_0(x). \quad (3.18)$$

Furthermore, for any $k \geq 0$, and $0 < t_0 < T$, $\phi \in C^\infty([t_0, T]; H^k(X; \mathbb{S} \otimes E))$. Thus, $\phi \in C^\infty([t_0, T] \times X; \mathbb{S} \otimes E)$.

Proof. It is clear that $\phi(t, x)$ defined in Eq. 3.18 solves Eq. 3.17 formally. Our first task will be to see that $\phi(t, x)$ is, in fact, well defined. We compute its L^2 -norm:

$$\|\phi(t, \cdot)\|_{L^2}^2 = \sum |(\phi, \psi_n)e^{-t\lambda_n}|^2 \leq \sum |(\phi, \psi_n)|^2 = \|\phi\|_{L^2}^2. \quad (3.19)$$

Thus the operator $\exp[-t\mathcal{D}_s^2] : L^2(X; \mathbb{S} \otimes E) \rightarrow L^2(X; \mathbb{S} \otimes E)$ is a bounded linear operator for any $t \geq 0$.

We now investigate the higher regularity of $\phi(t, \cdot)$. It suffices to bound $\|\mathcal{D}_s^k \phi\|_{L^2}$ for any $k \geq 0$ by virtue of Prop. 3.13 and the arguments that follow it. By the heat equation (Eq. 3.17)

$$\partial_t \phi(t, x) = -\mathcal{D}_s^2 \phi(t, x).$$

The L^2 -convergence of the sum in Eq. 3.18 is uniform for $t \geq t_0$, for any $t_0 > 0$, and one may thus interchange the summation and differentiation. We may thus compute $\partial_t \phi$ explicitly, and obtain

$$\|\mathcal{D}_s^2 \phi\|_{L^2}^2 \leq \sum |\lambda_n \phi_n e^{-t\lambda_n}|^2.$$

By repeated differentiation, we see that

$$\|\mathcal{D}_s^{2k} \phi\|_{L^2}^2 \leq \sum |p_k(\lambda_n) \phi_n e^{-t\lambda_n}|^2, \quad (3.20)$$

where $p_k(\lambda)$ is a polynomial. It is easy to check that the convergence of the sum in Eq. 3.19 implies the convergence of the sum in Eq. 3.20, and thus the existence of constants $C_{2k} > 0$ (for fixed s) such that

$$\|\mathcal{D}_s^{2k}\phi\|_{L^2} \leq C_{2k}\|\phi\|_{L^2}.$$

We thus see that $\exp[-t\mathcal{D}_s^2] : L^2(X; L^2(\mathbb{S} \otimes E)) \rightarrow L^2(X; H^k(\mathbb{S} \otimes E))$ is a bounded linear operator for any positive t , and any $k \geq 0$.

In order to get regularity in t , we note that Eq. 3.17 allows one to compute all time derivatives in terms of repeated application of \mathcal{D}_s^2 to $\phi(t, \cdot)$.

It remains to show uniqueness. Eq. 3.17 is linear, so that suffices to show that there is a unique solution with vanishing initial condition. We compute

$$\begin{aligned} \frac{d}{dt}\langle \phi, \phi \rangle_{L^2} &= \langle \partial_t \phi, \phi \rangle_{L^2} + \langle \phi, \partial_t \phi \rangle_{L^2} \\ &= -\langle \mathcal{D}_s^2 \phi, \phi \rangle_{L^2} - \langle \phi, \mathcal{D}_s^2 \phi \rangle_{L^2} \\ &= -2\langle \mathcal{D}_s \phi, \mathcal{D}_s \phi \rangle_{L^2} \\ &\leq 0. \end{aligned}$$

Thus the only solution with zero initial condition is the zero solution. \square

Prop. 3.16 may easily be extended to show that, for each fixed s , $\exp[-t\mathcal{D}_s^2]$ is a smoothing operator for positive time, that is, that for any $l \in \mathbb{Z}$, and $t > 0$ one has the bounded map $\exp[-t\mathcal{D}_s^2] : H^l(X; \mathbb{S} \otimes \nabla) \rightarrow C^\infty(X; \mathbb{S} \otimes \nabla)$. The operator thus has an integral kernel.

Proposition 3.21. *For each $s \neq 0$, there exists $p_{t,s}(x, y) : \mathbb{S} \otimes E_x \rightarrow \mathbb{S} \otimes E_y$, smooth in x , y , and t , such that*

1. $p_{t,s}(x, y) = p_{t,s}(y, x)^*$,
2. $\lim_{t \rightarrow 0} p_{t,s}(x, y) = \delta_x(y) \cdot \text{id}_{\mathbb{S} \otimes E}$, where the limit is taken in $H^l(X; \mathbb{S} \otimes E)$ for any $l < -\frac{1}{2} \dim(X)$,
- 3.

$$\exp[-t\mathcal{D}_s^2]\phi(x) = \int_X dy \, p_{t,s}(x, y)\phi(y).$$

Proof. Define

$$p_{t,s}(x, y) : \sigma \mapsto e^{-t\mathcal{D}^2}(\delta_\sigma)(x),$$

for $\sigma \in E_y$. This is clearly a linear map, and smoothness in x and t follows from Prop. 3.16. The fact that $p_{t,s}(x, y) = p_{t,s}(y, x)^*$ is an easy consequence of the formal self-adjointness of $e^{-t\mathcal{D}^2}$, and $p_{t,s}(x, y)$ is thus smooth in y .

We have

$$\lim_{t \rightarrow 0} e^{-t\mathcal{D}_s^2} \delta_\sigma = \delta_\sigma$$

in $H^l(X; \mathbb{S} \otimes E)$, for $l \leq -\frac{1}{2} \dim X$, giving

$$\lim_{t \rightarrow 0} p_{t,s}(x, y) = \delta_x(y) \cdot \text{id}_{\mathbb{S} \otimes E},$$

where the limit is taken in $H^l(X; \mathbb{S} \otimes E)$, $l \leq -\frac{1}{2} \dim X$.

Finally,

$$\begin{aligned}
\exp[-t\mathcal{D}_s^2]\phi &= \exp[-t\mathcal{D}_s^2] \int_X dy \delta_{\phi(y)}(x) \\
&= \int_X dy \exp[-t\mathcal{D}_s^2] \delta_{\phi(y)}(x) \\
&= \int_X dy p_{t,s}(x, y) \phi(y).
\end{aligned}$$

□

Remark 3.22. The arguments in [10] Ch. 2.7 carry through to show that the heat kernels $p_{t,s}(x, y)$ vary smoothly in s , for $s \neq 0$. In fact they show more: the heat kernels of any smooth family of Dirac operators on a compact manifold will vary smoothly.

Before we proceed with our investigation of the heat kernel, we give a basic estimate for smooth solutions to the heat equation.

Proposition 3.23. *Let $H_s = \frac{\partial}{\partial t} + \mathcal{D}_s^2$ be the heat operator associated to ∇ . There exist $C_l \in \mathbb{R}[|s|^{-1/2}, |s|^{1/2}]$ such that for any $\phi(x, t) \in C^\infty(X \times [0, \infty); \mathbb{S} \otimes E)$ and all $t \geq 0$*

$$\|\phi(\cdot, t)\|_{H^l} \leq C^l \left(\int_0^t du \|(H_s \phi)(\cdot, u)\|_{H^l} + \|\phi(\cdot, 0)\|_{H^l} \right).$$

Proof. We begin by noting that ϕ_t is a solution to

$$\begin{cases} H_s \psi = H_s \phi, \\ \psi(\cdot, 0) = \phi(\cdot, 0), \end{cases}$$

The heat equation with source $H\phi$, and initial value $\phi(\cdot, 0)$. By Duhamel's principle

$$\phi(x, t) = \int_0^t du e^{-(t-u)\mathcal{D}_s^2} (H\phi)(\cdot, u) + e^{t\mathcal{D}_s^2} \phi(\cdot, 0)$$

solves the equation. Thus, by uniqueness of solutions to the heat equation, $\psi = \phi$. We may thus estimate ψ by bounding the operator norm of $e^{-t\mathcal{D}_s^2} : H_l \rightarrow H_l$. For $l = 0$, the spectral decomposition of \mathcal{D}_s^2 implies $\|e^{-t\mathcal{D}_s^2}\|_{L^2} \leq 1$. For higher H^l , we use the estimate 3.13, and calculate

$$\begin{aligned}
\|e^{-t\mathcal{D}_s^2}f\|_{H^l} &\leq C_s \left(\|\mathcal{D}_s^l e^{-t\mathcal{D}_s^2}f\|_{L^2} + \|f\|_{L^2} \right) \\
&= C_s \left(\|e^{-t\mathcal{D}_s^2}\mathcal{D}_s^l f\|_{L^2} + \|f\|_{L^2} \right) \\
&\leq C_s (\|\mathcal{D}_s^l f\|_{L^2} + \|f\|_{L^2}) \\
&\leq C_s \|f\|_{H^l}.
\end{aligned}$$

Note that the constant depends rationally on $|s|^{1/2}$ precisely because the constants in Eq. 3.13 and \mathcal{D}_s^l both have rational dependence on $|s|^{1/2}$. \square

Let us now try to develop some understanding of the heat kernel. We may think of the heat kernel $p_{t,s}(x, y)$ as telling us how much heat has flowed from x to y in time t , when the heat was all originally concentrated at x . It is reasonable to expect that when $x \neq y$, $p_{t,s}(x, y)$ decays to zero as t approaches zero. This is the content of the next proposition.

Proposition 3.24. *For any small enough $d > 0$, there exists $c > 0$, and positive $C_s \in \mathbb{R}[|s|^{-1/2}, |s|^{1/2}]$ such that*

$$\|p_{t,s}(x, y)\|_{C^k} \leq C_s e^{-c/t}$$

for $\text{dist}(x, y) > d$ and $t > 0$. The constant c may be chosen less than $d^2/4$; C_s then depends only the superconnection, c and k .

Remark 3.25. The dependence on s in the Prop. 3.24 is crucial. The key point is that the speed of propagation in Prop. 3.28 is independent of the choice of superconnection.

In order to prove Prop. 3.24, we will investigate a transport equations associated with the Dirac operator:

$$\frac{\partial}{\partial t}\psi = i\mathcal{D}_s\psi, \quad (3.26)$$

where, as usual ψ is some time varying section of $\mathbb{S} \otimes E$. This equation may be seen as one factor in a factorisation of the wave equation associated with the laplacian.

The properties of 3.26 are very similar to those of the classical wave equation on \mathbb{R}^n . In particular, the equation exhibits no smoothing, and has finite speed of propagation. We will address the second property shortly, but before doing so, let us write solutions in terms of an exponential of the Dirac operator.

Proposition 3.27. *There is a bounded operator $\exp[it\mathcal{D}_s] : H^l(X; \mathbb{S} \otimes E) \rightarrow H^l(X; \mathbb{S} \otimes E)$ for every t and l . Thus*

$$e^{it\mathcal{D}_s}\phi$$

gives the unique solution to Eq. 3.26 with initial condition $\phi \in H^l(X; \mathbb{S} \otimes E)$.

Furthermore,

$$\|e^{it\mathcal{D}_s}\phi\|_{L^2} = \|\phi\|_{L^2}.$$

The solution is smooth if the initial condition is.

Proof. The proposition may easily be proved along the lines of Prop. 3.16. To show

$$\|e^{it\mathcal{D}_s}\phi\|_{L^2} = \|\phi\|_{L^2}$$

one computes

$$\begin{aligned} \|e^{it\mathcal{D}_s}\phi\|_{L^2}^2 &= \langle e^{it\mathcal{D}_s}\phi, e^{it\mathcal{D}_s}\phi \rangle_{L^2} \\ &= \langle e^{-it\mathcal{D}_s}e^{it\mathcal{D}_s}\phi, \phi \rangle_{L^2} \\ &= \langle \phi, \phi \rangle_{L^2} \\ &= \|\phi\|_{L^2}^2. \end{aligned}$$

□

The next proposition makes the notion of finite speed of propagation for the transport equation precise. It is the key to the uniformity remarked upon in Rem. 3.25: the speed of propagation is *independent of the choice of unitary superconnection*.

Proposition 3.28. *Let $K \subset X$ be compact, and suppose $\phi \in C^\infty(X, \mathbb{S} \otimes E)$ has support $\text{supp } \phi \subset K$. Define $\phi_t = \exp it\mathcal{D}(\nabla)_s\phi$. Then, for sufficiently small $|t|$,*

$$\text{supp } \phi_t \subseteq \{x \in X : \text{dist}(x, K) \leq |t|\}.$$

This proposition is independent of the choice of unitary superconnection.

Proof. Let $\phi(x, t)$ be a smooth solution to the transport equation. We wish to estimate the energy of ϕ inside a ball $B_R(x)$, where R is less than the injectivity

radius of X , and $x \in X$. It will be convenient to define the one-form η : for $\xi \in \Gamma(T(X))$, $\eta(t) : \xi \mapsto \langle \xi \cdot \phi(\cdot, t), \phi(\cdot, t) \rangle_E$. We have

$$\begin{aligned}
\frac{d}{dt} \int_{B_R(x)} dy \|\phi(y, t)\|_E^2 &= i \int_{B_R(x)} dy \langle \mathcal{D}_s \phi(y, t), \phi(y, t) \rangle_E \\
&\quad - \langle \phi(y, t), \mathcal{D}_s \phi(y, t) \rangle_E \\
&= i \int_{B_R(x)} dy \langle \mathcal{D} \phi(y, t), \phi(y, t) \rangle_E \\
&\quad - \langle \phi(y, t), \mathcal{D} \phi(y, t) \rangle_E \\
&= -i \int_{B_R(x)} dy d^* \eta(t) \\
&= i \int_{\partial B_R(x)} dy \eta(t) \cdot \nu \\
&= i \int_{\partial B_R(x)} dy \langle \nu \cdot \phi(y, t), \phi(y, t) \rangle_E, \quad (3.29)
\end{aligned}$$

where ν is an outward pointing normal vector field on $\partial B_R(x)$ and the second line follows from the unitarity of \mathcal{E} . It is this step that is the key to establishing the uniformity in the proposition. We now have the estimate

$$\frac{d}{dt} \int_{B_{R-|t|}(x)} dy \|\phi(y, t)\|_E^2 \leq \int_{\partial B_{R-|t|}(x)} dy \langle i\nu \cdot \phi(y, t) - \phi(y, t), \phi(y, t) \rangle_E \leq 0. \quad (3.30)$$

Having established the estimate 3.30, we may now prove the proposition. Let $\phi(x, t)$ be a smooth solution to Eq. 3.26 with initial condition $\phi_0(x)$. Choose x, R , such that $B_R(x) \cap \text{supp}(\phi_0) = \emptyset$. Then

$$\int_{B_R(x)} dy \|\phi(y, 0)\|_E^2 = 0,$$

so that, by the Eq. 3.30,

$$\int_{B_{R-|t|}(x)} dy \|\phi(y, t)\|_E^2 = 0,$$

for any $t < R$. This proves the theorem for smooth solutions. For solutions in $H^l(X; \mathbb{S} \otimes E)$, we approximate by smooth functions. \square

The next lemma tells us how we may use the transport equation to give us information about the heat equation.

Lemma 3.31. *We have*

$$\exp[-t\mathcal{D}_s^2] = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} du e^{-u^2/4t} \exp[iu\mathcal{D}_s]. \quad (3.32)$$

Proof. We begin by proving the following identity:

$$e^{-\lambda t} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} du e^{-u^2/4t} e^{\pm iu\sqrt{\lambda}}.$$

Completing the square on the left hand side we see

$$\int_{-\infty}^{\infty} du e^{-u^2/4t} e^{\pm iu\sqrt{\lambda}} = e^{-t\lambda} \int_{-\infty}^{\infty} du e^{-(u/\sqrt{4t} \mp i\sqrt{t}\lambda)^2}.$$

Changing variables in the second integral reduces it to the standard Gaussian integral, proving the identity. Decomposing 3.32 in eigenspaces of \mathcal{D}_s^2 and applying this identity gives the result. \square

We are finally in a position to prove Prop. 3.24.

Proof of 3.24. Choose $d \in \mathbb{R}^{\geq 0}$, and $x, y \in X$ such that $\text{dist}(x, y) > d$. Let $c < d^2/4$. We will show that for any $\sigma \in E_y$ of unit norm

$$\|p_{t,s}(x, y)\sigma\|_{H^l} \leq C_s e^{-c/t},$$

for some fixed $C \in \mathbb{R}[|s|^{-1/2}, |s|^{1/2}]$. Eq. 3.32 and Prop. 3.24 allow us to write

$$p_{t,s}(x, y)\sigma = \frac{1}{\sqrt{4\pi t}} \int_{|u|>d} du e^{-u^2/4t} (e^{iu\mathcal{D}_s} \delta_\sigma)(x).$$

It is sufficient to bound the $L^2(X; \mathbb{S} \otimes E)$ operator norm of

$$\psi \mapsto \frac{1}{\sqrt{4\pi t}} \int_{|u|>d} du e^{-(u^2/4-c)/t} \mathcal{D}_s^k e^{iu\mathcal{D}_s} \psi.$$

We do this with the spectral decomposition. On the λ eigenspace

$$\begin{aligned} \left| \frac{1}{\sqrt{4\pi t}} \int_{|u|>d} du e^{-(u^2/4-c)/t} (\pm \lambda^{k/2}) e^{\pm iu\sqrt{\lambda}} \right| &\leq \frac{C e^{-\varepsilon/t}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} du \lambda^{k/2} e^{-u^2/4t + iu\sqrt{\lambda}} \\ &= C_0 e^{-\varepsilon/t} e^{-t\lambda} \lambda^{k/2} \\ &\leq C_0. \end{aligned}$$

Prop. 3.13 completes the estimate. Note that by virtue of the fact that C_0 is constant, and the coefficients in Prop. 3.13 are rational functions in $|s|^{1/2}$ we have $C_s \in \mathbb{R}[|s|^{-1/2}, |s|^{1/2}]$. \square

Now that we understand the behaviour of the heat kernel away from the diagonal, we must turn our attention to the diagonal.

3.3.1 The heat kernel on the diagonal

Up until this point we have been examining the behaviour of the heat kernel, treating s as a free parameter. We now wish to couple t and s , and examine the small time behaviour of $p_{t,s/t}(x, x)$, with s fixed. We will see that $p_{t,s/t}(x, x)$ has a small t asymptotic expansion, with the leading term of order $t^{-\dim X/2}$. The coupling of the parameters fundamentally new ingredient in

considering Dirac operators coupled to superconnections. The usual derivations of the small t asymptotic expansion of the heat kernel are difficult to extend to find a small time asymptotic expansion for the coupled parameters. Getzler does so in [22] using the Feynman-Kac formula. We do so extending a geometric argument found in [17].

3.3.2 An asymptotic expansion for the heat kernel

We will show that $p_{t,s/t}(x, x)$ has a small t asymptotic expansion with local coefficients and leading order $t^{-n/2}$, where $n = \dim X$. This will be done in a rather indirect manner: we will use parabolic scaling to relate small t behaviour to the behaviour of the family of heat kernels under blowup, and examine this behaviour. In order to do so, we will cut a neighbourhood of the point of interest out of the manifold, and paste it into \mathbb{R}^n , and use our knowledge of parabolic equations on \mathbb{R}^n to show the existence of the asymptotic expansion. This approach is essentially found in [17], but instead of working with a family of compact manifolds with a noncompact fibre, we find it more convenient to relate the heat kernel on X to one on \mathbb{R}^n , and work there.

Let $x \in X$, and $r > 0$ be such that

$$\exp : B_r(0) \rightarrow U$$

be a diffeomorphism, where $B_r(x) \subset T_x(X)$, and $U \subset X$. Choose a monotone function $\rho \in C^\infty(\mathbb{R})$ such that $\rho(x) = 0$ for $x > r$, and $\rho(x) = 1$ for $x < 3r/4$. Pull back the geometry on X to $B_r(0)$ with the exponential map and obtain a

complex, hermitian \mathbb{Z}_2 -graded vector bundle $E \rightarrow B_r(0)$, with superconnection ∇^s , and a metric g and bundle of spinors $\mathbb{S} \rightarrow B_r(0)$ (we abuse notation slightly, suppressing the \exp^*). We trivialise $E \rightarrow B_r(0)$ and $\mathbb{S} \rightarrow B_r(0)$ by parallel transport along radial geodesics. We extend E and S trivially to all of $T_x(X)$, and extend the superconnection and metric using ρ , such that on the complement of $B_r(0)$ the geometry is constant (we set it equal to that at the origin). Precisely, define

$$g = \rho \exp^* g + (1 - \rho)g(0),$$

and ∇^s similarly. We now have put a metric on $T_x(X)$, and endowed it with trivial, complex and hermitian \mathbb{Z}_2 -graded vector bundles $E, \mathbb{S} \rightarrow T_x(X)$, with E endowed with the superconnection ∇^s .

We define the operators $T_\varepsilon : T_x(X) \rightarrow T_x(X)$ by $T_\varepsilon(\xi) = \varepsilon\xi$. With the help of these operators, we define the family of heat operators

$$H_\varepsilon = \frac{\partial}{\partial t} + \varepsilon^2 T_\varepsilon^* \mathcal{D} \left(\nabla^{s/\varepsilon^2} \right)^2 (T_\varepsilon^*)^{-1} \quad (3.33)$$

with heat kernels $p_{t,s}^\varepsilon(\xi, \xi')$. The next lemma relates the heat kernels $p_{t,s}^\varepsilon(0, \xi)$ to $\exp^* p_{t,s}(0, \xi)$.

Proposition 3.34. *For any $r' < 3r/4$, there exist a constant c , and rational function $C(t) \in \mathbb{R}(t)$ such that for $|\xi| < r'$, and $|\varepsilon|$ small enough,*

$$\left| \exp^* p_{t,s}(0, \xi) - |\varepsilon|^{-n} p_{t/\varepsilon^2, \varepsilon^2 s}^\varepsilon(0, \xi/\varepsilon) \right| \leq C(|\varepsilon|) e^{-c/t}.$$

In fact, c may be chosen to be $r'^2/4$.

Straightforward computation yields the following lemma.

Lemma 3.35. *Let $\rho \in C^\infty(X)$, and $\psi \in \Gamma(\mathbb{S} \otimes E)$. Then*

$$\mathcal{D}(\nabla^s)^2(\rho\psi) = c(\nabla^2\rho)\psi - \langle d\rho, \nabla\psi \rangle + \sum_{i \geq 2} [c(\omega_s^i), c(d\rho)]\phi + \rho \mathcal{D}(\nabla^s)^2\psi,$$

where $\nabla^s = \nabla + \sum_{i \geq 0} \omega_s^i$, $\omega_s^i \in \Omega^i(X; \text{End}(E))$.

With this lemma in place, we may prove Prop. 3.34.

Proof of 3.34. We fix $x \in X$. Let $\delta = (3r/4 - r')/4$ and $\rho \in C^\infty(X)$ be a monotone function such that $\rho(y) = 1$ for $\text{dist}(x, y) < r'$, and $\rho(y) = 0$ for $\text{dist}(x, y) > 3r/4 - \delta$. We define ϕ_t on X as follows

$$\phi_t(y) = \begin{cases} p_{t,s}(x, y) - \rho(y)\varepsilon^{-n}p_{t/\varepsilon^2, \varepsilon^2s}^\varepsilon(y/\varepsilon) & y \in U, \\ p_{t,s}(x, y) & y \in X \setminus U. \end{cases}$$

Here, and for the remainder of the proof, we implicitly identify points in U with points in $B_r(0) \subset T_x(X)$ using the exponential map, and thus make sense of expressions like y/ε for $y \in U$ by seeing $y \in B_r(0)$. We wish to estimate the C^k -norm of $H\phi_t$, where

$$H = \frac{\partial}{\partial t} + \mathcal{D}(\nabla^s)^2$$

is the heat operator associated with ∇^s . By definition

$$Hp_{t,s}(x, y) = 0,$$

and a short computation shows

$$Hp_{t/\varepsilon^2, \varepsilon^2s}^\varepsilon(x, y/\varepsilon) = 0,$$

for $y \in U$. Thus the only contribution to $H\phi_t$ comes from

$$H\left(\rho(y)\varepsilon^{-n}p_{t/\varepsilon^2,\varepsilon^2s}^\varepsilon(x,y/\varepsilon)\right)$$

for y such that $r' < \text{dist}(x,y) < 3r/4 - \delta$. By lemma 3.34,

$$\begin{aligned} H\phi_t(y) = -\varepsilon^{-n} & \left[c(\nabla^2 \rho) p_{t/\varepsilon^2,\varepsilon^2s}^\varepsilon(x,y/\varepsilon^2) - 2 \left\langle d\rho, \nabla p_{t/\varepsilon^2,\varepsilon^2s}^\varepsilon(x,y/\varepsilon^2) \right\rangle \right. \\ & \left. + \sum_{i \geq 2} [c(\omega_i^s), c(d\rho)] p_{t/\varepsilon^2,\varepsilon^2s}^\varepsilon(x,y/\varepsilon^2) \right]. \end{aligned}$$

The C^k norms of ρ, ω^s are bounded, and by Prop. 3.24,

$$\left\| p_{t/\varepsilon^2,s\varepsilon^2}^\varepsilon(x,y/\varepsilon) \right\|_{C^k} \leq C_{s\varepsilon^2} e^{-r'^2/4t}.$$

Combining these estimates shows

$$\|H\phi_t\|_{C^k} \leq C_{s\varepsilon^2} |\varepsilon|^{-n} e^{-r'^2/4t}.$$

We now wish to show $\lim_{t \rightarrow 0} \phi_t = 0$. Both $p_{t,s}^\varepsilon(x,y)$ and $\varepsilon^{-n} p_{t/\varepsilon^2,s\varepsilon^2}^\varepsilon(x,y/\varepsilon)$ converge to $\delta_x \text{id}_{E_x}$ in H^l , $l < -\dim X/2$ as $t \rightarrow 0$, so that

$$\lim_{t \rightarrow 0} \phi_t = 0$$

in H^l , $l < -\dim X/2$. We define

$$\psi_t = - \int_0^t du e^{-(u-t)\mathcal{D}(\nabla^s)^2} (H\phi)_u,$$

and note that it is smooth, and converges to zero in C^∞ as $t \rightarrow 0$. But $H(\phi_t + \psi_t) = 0$, and thus, by uniqueness of solutions to the heat equation, and $\lim_{t \rightarrow 0} (\phi_t + \psi_t) = 0$ in H^l , $l < -\dim X/2$, we conclude

$$\psi_t = -\phi_t.$$

But then ϕ_t converges to zero as $t \rightarrow 0$ in C^∞ , as asserted.

Finally, the basic estimate 3.23 shows

$$\|\phi_t\|_{H^l} \leq C_{s\varepsilon^2} |\varepsilon|^{-n} e^{-r'^2/4t},$$

where $C_{s\varepsilon^2}$ is a rational function in $|\varepsilon|$ with degree depending on l . Choosing l large enough and using the Sobolev embedding theorem we obtain the result. \square

We are finally in a position to show $p_{t,s/t}(x, y)$ has a small time asymptotic expansion with local coefficients. To do so, we note that the family of heat equations 3.33 fits the hypotheses of Prop. 3.56 in Sec. 3.4 and we conclude that, for t bounded away from 0, $p_{t,s}^\varepsilon(x, y)$ varies smoothly in all of its variables. We may thus take the epsilon Taylor expansion

$$p_{1,s}^\varepsilon(0, 0) = \sum_{k=0}^N b_k \varepsilon^k + O(|\varepsilon|^{N+1}),$$

holding all other variables constant. The b_k are smooth in u by virtue of the fact that $p_{t,s}^\varepsilon(x, y)$ varies smoothly in s . The b^i are local in the geometry of X : indeed, all ε derivatives of H^ε at $\varepsilon = 0$ have coefficients depending only on the jet of the geometry at x . Prop. 3.34 applied twice shows that

$$|p_{1,s}^\varepsilon(0, 0) - p_{1,s}^{-\varepsilon}(0, 0)| < C(|\varepsilon|) e^{-c/\varepsilon^2},$$

allowing us to conclude that that up to an that vanishes to all orders of ε , $p_{1,u}^\varepsilon(0, 0)$ is even in ε , and the coefficients b_{2l+1} all vanish. Prop. 3.34 also

shows that

$$\exp^* p_{t,s/t}(0,0) = \varepsilon^{-n} p_{t/\varepsilon^2, s\varepsilon^2/t}^\varepsilon(0,0) + O\left(C(|\varepsilon|)e^{-c/t}\right),$$

so that, setting $\varepsilon = \sqrt{t}$, one has

$$\exp^* p_{t,s/t}(0,0) = t^{-n/2} p_{1,s}^{\sqrt{t}}(0,0) + O\left(C(\sqrt{t})e^{-c/t}\right),$$

or, setting

$$a_k = (4\pi)^{-n/2} b_{2k},$$

one arrives at

$$\exp^* p_{t,s/t}(0,0) = (4\pi t)^{-n/2} \sum_{k \geq 0}^N a_k t^k + O\left(t^{N-n/2+1/2-\varepsilon}\right),$$

for ε arbitrarily small.

Proposition 3.36. *For any $x \in X$, the heat kernel $p_{t,s/t}(x,x)$ has an asymptotic expansion of the form*

$$p_{t,s/t}(x,x) = (4\pi t)^{-n/2} \sum_{k \geq 0}^N a_k(x,s) t^k + O\left(t^{N-n/2+1/2}\right),$$

where the $a_k(x,s) \in \text{Cl}(X)_x \otimes \text{End}(E_x)$ are local, and vary smoothly in x and s .

3.4 The kernel of a smooth family of heat operators

We wish to examine a particular family of parabolic linear PDEs on \mathbb{R}^n . Let

$$\Delta = - \sum_i \frac{\partial^2}{\partial x^{i^2}}$$

be the positive definite laplacian on \mathbb{R}^n . Let

$$L_\varepsilon = \Delta + f^i(\varepsilon, x) \frac{\partial}{\partial x^i} + g(\varepsilon, x) \quad (3.37)$$

be a formally self-adjoint elliptic operator on functions on \mathbb{R}^n with values in V , a finite dimensional vector space, and let the f^i be of the form

$$f^i(\varepsilon, x) = \sum_{j=0}^k \varepsilon^j f_j^i(\varepsilon x), \quad (3.38)$$

with $f_j^i(x) \in C^\infty(\mathbb{R}^n; \text{End}(V))$ and constant outside a compact set, and $g(\varepsilon, x)$ similar. Here we use the usual summation convention, where repeated indices are summed over. We shall examine the family of heat operators

$$H_\varepsilon = \frac{\partial}{\partial t} + L_\varepsilon. \quad (3.39)$$

We first show the existence of integral kernels for the semigroups associated to these operators.

Proposition 3.40. *For every $\varepsilon \in \mathbb{R}$, the semigroup on $L^2(\mathbb{R}^n; V)$ associated to H_ε has an integral kernel $k_\varepsilon(t; x, y)$. The integral kernel is smooth, and in $H^k(\mathbb{R}^n; \text{End}(V))$ for every $k \in \mathbb{Z}$.*

Proof. We begin by recalling that the integral kernel associated to the standard heat semigroup on \mathbb{R}^n is

$$k(t; x) = (4\pi t)^{-n/2} \exp\left(-\frac{\|x\|^2}{4t}\right), \quad (3.41)$$

and its Fourier transform is given by

$$\hat{k}(t, \xi) = (2\pi)^{n/2} e^{-t\|\xi\|^2}. \quad (3.42)$$

One may use the Volterra series to give a formal expression for the semigroup associated to H_ε in terms of the standard heat semigroup:

$$e^{-tL_\varepsilon} = \sum_{k \geq 0} I_k(t, \varepsilon), \quad (3.43)$$

where

$$I_k(t, \varepsilon) = - \int_0^t d\sigma e^{-(t-\sigma)\Delta} \left(f^i(\varepsilon, x) \frac{\partial}{\partial x^i} + g(\varepsilon, x) \right) I_{k-1}(\sigma, \varepsilon) \quad (3.44)$$

and

$$I_0(s) = e^{-s\Delta}. \quad (3.45)$$

In terms of Eq. 3.44, the integral kernel associated to the heat semigroup is defined by

$$k_\varepsilon(t; x, y) = e^{-tL_\varepsilon} \delta_y(x) = \sum_{k \geq 0} I_k(t, \varepsilon) \delta_y(x). \quad (3.46)$$

We will show that

$$k_\varepsilon(t; \cdot, 0) = \sum_{k \geq 0} I_k(t, \varepsilon) \delta_0 \quad (3.47)$$

converges absolutely in $H^k(\mathbb{R}^n; V)$ for all $k \geq 0$ as long as $t > 0$, and thus, for fixed t , $k_\varepsilon(t; \cdot, 0) \in C^\infty(\mathbb{R}^n; V)$ (by the Sobolev lemma).

We begin by defining integral kernels for the I_k :

$$J_k(t, \varepsilon; \cdot) = I_k(t, \varepsilon) \delta_0. \quad (3.48)$$

Our first task will be to bound the Fourier transforms of the J_k pointwise. A short computation gives the following lemma.

Lemma 3.49. *The Fourier transforms of $f^i(\varepsilon, x) \frac{\partial}{\partial x^i} k(t; x)$ and $g(\varepsilon, x) k(t; \varepsilon)$ are bounded by:*

$$\left\| \mathcal{F} \left[f^i(\varepsilon, \cdot) \frac{\partial}{\partial x^i} k(t; \cdot) \right] (\xi) \right\|_V \leq C_0 \|\xi\| \hat{k}(t; \xi), \quad (3.50)$$

$$\left\| \mathcal{F} [g(\varepsilon, \cdot) k(t; \cdot)] (\xi) \right\|_V \leq C_1 \hat{k}(t; \xi), \quad (3.51)$$

where $\|\cdot\|_V$ is the norm on V , and $\mathcal{F}[\cdot]$ denotes the Fourier transform.

We now estimate the Fourier transform of $J_k(s, \varepsilon; \cdot)$.

Lemma 3.52. *The Fourier transform of $J_k(s, \varepsilon; \cdot)$ has the pointwise bound*

$$\left\| \hat{J}_k(t, \varepsilon; \xi) \right\|_V \leq \frac{(\|\xi\| C_0 + C_1)^k t^k}{k!} \hat{k}(t; \xi),$$

where

$$\hat{J}_k(t, \varepsilon; \cdot) = \mathcal{F}[J_k(t, \varepsilon; \cdot)].$$

Proof. We begin with an non-inductive formula for the J_k in terms of the standard heat kernel. By definition (3.48, 3.44),

$$J_k(t, \varepsilon; \cdot) = - \int_0^t d\sigma k(t - \sigma; \cdot) * \left[\left(f^i(\varepsilon, \cdot) \frac{\partial}{\partial x^i} + g(\varepsilon, \cdot) \right) J_{k-1}(\sigma, \varepsilon; \cdot) \right],$$

where “ $*$ ” denotes convolution on \mathbb{R}^n , and

$$J_0(t, \varepsilon; \cdot) = k(t; \cdot).$$

But then

$$\begin{aligned} J_k(t, \varepsilon; \cdot) = - \int_{t_{\Delta_k}} d\sigma k(\sigma_0; \cdot) * & \left[\left(f^i(\varepsilon, \cdot) \frac{\partial}{\partial x^i} + g(\varepsilon, \cdot) \right) k(\sigma_1; \cdot) \right] * \dots \\ & \dots * \left[\left(f^i(\varepsilon, \cdot) \frac{\partial}{\partial x^i} + g(\varepsilon, \cdot) \right) k(\sigma_k; \cdot) \right], \end{aligned} \quad (3.53)$$

where

$$\Delta_k = \left\{ \boldsymbol{\sigma} \in \mathbb{R}^{k+1} : \sum_i \sigma_i = 1 \right\}.$$

Thus

$$\begin{aligned} \hat{J}_k(t, \varepsilon; \cdot) = & - \int_{s\Delta_k} d\boldsymbol{\sigma} \hat{k}(\sigma_0, \xi) \mathcal{F} \left[\left(f^i(\varepsilon, \cdot) \frac{\partial}{\partial x^i} + g(\varepsilon, \cdot) \right) k(\sigma_1; \cdot) \right] \dots \\ & \dots \mathcal{F} \left[\left(f^i(\varepsilon, \cdot) \frac{\partial}{\partial x^i} + g(\varepsilon, \cdot) \right) k(\sigma_k; \cdot) \right]. \end{aligned} \quad (3.54)$$

Lemma 3.49 shows that

$$\left\| \mathcal{F} \left[\left(f^i(\varepsilon, \cdot) \frac{\partial}{\partial x^i} + g(\varepsilon, \cdot) \right) k(t; \cdot) \right] (\xi) \right\|_V \leq (C_0 \|\xi\| + C_1) \hat{k}(t; \xi).$$

Thus

$$\begin{aligned} \left\| \hat{J}_k(t, \varepsilon; \xi) \right\|_V & \leq \int_{t\Delta_k} d\boldsymbol{\sigma} \hat{k}(\sigma_0, \xi) \left\| \mathcal{F} \left[\left(f^i(\varepsilon, \cdot) \frac{\partial}{\partial x^i} + g(\varepsilon, \cdot) \right) k(\sigma_1; \cdot) \right] (\xi) \right\|_V \times \\ & \quad \dots \left\| \mathcal{F} \left[\left(f^i(\varepsilon, \cdot) \frac{\partial}{\partial x^i} + g(\varepsilon, \cdot) \right) k(\sigma_k; \cdot) \right] (\xi) \right\|_V \\ & \leq \int_{t\Delta_k} d\boldsymbol{\sigma} (C_0 \|\xi\| + C_1)^k \hat{k}(\sigma_0; \xi) \hat{k}(\sigma_1; \xi) \dots \hat{k}(\sigma_k; \xi) \\ & \leq (C_0 \|\xi\| + C_1)^k \hat{k}(t, \xi) \text{vol}(t\Delta_k) \\ & \leq \frac{(C_0 \|\xi\| + C_1)^k t^k}{k!} \hat{k}(t; \xi). \end{aligned}$$

□

We may now use 3.52 to bound the Fourier transform of $k_\varepsilon(t; \cdot, 0)$:

$$\begin{aligned} \left\| \mathcal{F} [k_\varepsilon(t; \cdot, 0)] (\xi) \right\|_V & \leq \sum_{k \geq 0} \left\| \hat{J}_k(t, \varepsilon; \xi) \right\|_V \\ & \leq \sum_{k \geq 0} \frac{(C_0 \|\xi\| + C_1)^k t^k}{k!} \hat{k}(t; \xi) \\ & \leq (2\pi)^n e^{-t(\|\xi\|^2 - C_0 \|\xi\| - C_1)} \end{aligned} \quad (3.55)$$

The Fourier transform of $k_\varepsilon(t; \cdot, 0)$ is thus dominated by a shifted Gaussian, and so $k_\varepsilon(t; \cdot, 0) \in H^k(\mathbb{R}^n; V)$ for any $k \in \mathbb{Z}$ and $t > 0$. The Sobolev lemma thus shows that $k_\varepsilon(t; \cdot, 0)$ is smooth, as claimed. Exchanging δ_0 with δ_y in Eq. 3.47 and doing similar analysis shows $k_\varepsilon(t; \cdot, y)$ is smooth. To show smoothness in y it suffices to note that by the formal self-adjointness of L_ε the function $k_\varepsilon(t; x, y)$ is symmetric in x and y . Smoothness in t is immediate, as the kernel satisfies $H_\varepsilon k_\varepsilon(\cdot; \cdot, y) = 0$, allowing one to compute all time derivatives in terms of spacial derivatives of $k_\varepsilon(t; x, y)$. \square

Now that we have established the existence and regularity of the integral kernel for the semigroup associated with H_ε for every ε , we wish to show smoothness in ε .

Proposition 3.56. *The integral kernels $k_\varepsilon(t; x, y)$ vary smoothly in ε .*

Proof. We will only show smoothness at $\varepsilon = 0$. Smoothness at non-zero ε is easy to show as the ε derivatives of the f^i, g vanish outside a compact set for $\varepsilon \neq 0$. In order to simplify our presentation, we shall henceforth restrict our attention to the “degree zero” parts of $f^i(\varepsilon, x), g(\varepsilon, x)$ and assume

$$\begin{aligned} f^i(x, \varepsilon) &= f^i(\varepsilon x), \\ g^i(x, \varepsilon) &= g^i(\varepsilon x). \end{aligned}$$

It will be obvious that including the higher degree terms in Eq. 3.38 does not affect the argument substantially.

We begin with a simple lemma.

Lemma 3.57. *Let $\varepsilon \in [-1, 1]$, and $f(x) \in C^\infty(\mathbb{R}^n)$ and bounded. Then there exists $C > 0$ such that*

$$\left| \frac{f(\varepsilon x) - f(0)}{\varepsilon} \right| \leq C\|x\|.$$

Proof. One has

$$\frac{f(\varepsilon x) - f(0)}{\varepsilon} = \|x\| \frac{f(\varepsilon x) - f(0)}{\varepsilon\|x\|}.$$

The quotient

$$\frac{f(\varepsilon x) - f(0)}{\varepsilon\|x\|}$$

is continuous in ε , as f is smooth. Thus there exists a constant such that

$$\left| \frac{f(\varepsilon) - f(0)}{\varepsilon} \right| < C_0$$

for $\varepsilon < 1$. On the other hand, the boundedness of f shows

$$\left| \frac{f(\varepsilon) - f(0)}{\varepsilon} \right| < C_1$$

for $\varepsilon \geq 1$. Combining these estimates gives a uniform bound

$$\left| \frac{f(\varepsilon x) - f(0)}{\varepsilon\|x\|} \right| < C,$$

and the result. □

We would like to differentiate the $J_k(t, \varepsilon; x)$ (3.48) under the integral. Applying lemma 3.57 repeatedly, and recalling the boundedness of the f^i and g shows the integrand in the difference quotient

$$\frac{J_k(t, \varepsilon; x) - J_k(t, 0; x)}{\varepsilon} \tag{3.58}$$

to be dominated (pointwise) by a function of the form

$$Ck(\sigma_0; x) * ((\|x\| + 1)(\nabla + 1)k(\sigma_1; x)) * \dots * ((\|x\| + 1)(\nabla + 1)k(\sigma_k; x)) \quad (3.59)$$

where C is a constant, and $\varepsilon \in [-1, 1]$, and convolution is taken in \mathbb{R}^n . The gaussian decay of $k(t; x)$ ensures that the function in Eq. 3.59 is integrable, so that we may apply the dominated convergence theorem to commute the derivative past the integral in computing

$$\left. \frac{\partial}{\partial \varepsilon} J_k(t, \varepsilon; x) \right|_{\varepsilon=0}.$$

The chain rule shows

$$\left. \frac{\partial}{\partial \varepsilon} \left(\left(f^i(\varepsilon x) \frac{\partial}{\partial x^i} + g(\varepsilon x) \right) k(t; x) \right) \right|_{\varepsilon=0} = \left(x^j (\nabla_j f^i)(0) \frac{\partial}{\partial x^i} + x^i \nabla_i g(0) \right) k(t; x). \quad (3.60)$$

We compute the Fourier transform of Eq. 3.60:

$$\begin{aligned} \mathcal{F} \left[\left. \frac{\partial}{\partial \varepsilon} \left(f^i(\varepsilon x) \frac{\partial}{\partial x^i} + g(\varepsilon x) \right) k(t; x) \right|_{\varepsilon=0} \right] (\xi) \\ = (i(\nabla_j f^i)(0) \xi^i + \nabla_j g(0)) \nabla_j \hat{k}(t; \xi) + i \nabla_i f^i(0) \hat{k}(t; \xi). \end{aligned}$$

Computing $\nabla_i \hat{k}(t; \xi) = -2t \xi^i \hat{k}(t; \xi)$, one has

$$\begin{aligned} \mathcal{F} \left[\left. \frac{\partial}{\partial \varepsilon} \left(f^i(\varepsilon x) \frac{\partial}{\partial x^i} + g(\varepsilon x) \right) k(t; x) \right|_{\varepsilon=0} \right] (\xi) \\ = [-2ti (i(\nabla_j f^i)(0) \xi^i + \nabla_j g(0)) \xi^j - \nabla_i f^i(0)] \hat{k}(t; \xi). \end{aligned}$$

On the other hand, a short computation shows

$$\mathcal{F} \left[\left(f^i(0) \frac{\partial}{\partial x^i} + g(0) \right) k(t; x) \right] (\xi) = (i f^i(0) \xi^i + g(0)) \hat{k}(t; \xi).$$

Putting this together (and recalling the form of the \hat{J}_k given in Eq. 3.54), one computes

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} \hat{J}_k(\varepsilon, t; \xi) = & - \left[-2t \left(i(\nabla_j f^i)(0) \xi^i + \nabla_j g(0) \right) \xi^j i + \right. \\ & \left. + i \nabla_i f^i(0) \right] \frac{(i f^i(0) \xi^i + g(0))^{k-1} t^k}{(k-1)!} \hat{k}(t; \xi). \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} \mathcal{F}[k_\varepsilon(t; \cdot, 0)](\xi) \Big|_{\varepsilon=0} = & \left[-2ti \left(i(\nabla_j f^i)(0) \xi^i + \nabla_j g(0) \right) \xi^j \right. \\ & \left. - \nabla_i f^i(0) \right] t e^{-(i f^i(0) \xi^i + g(0))t} \hat{k}(t; \xi). \quad (3.61) \end{aligned}$$

Inverting the Fourier transform in Eq. 3.61, we obtain the explicit expression:

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} k_\varepsilon(t; x) \Big|_{\varepsilon=0} = & \left[2t \left((\nabla_j f^i)(0) \frac{\partial^2}{\partial x^i \partial x^j} + (\nabla_j g)(0) \frac{\partial}{\partial x^j} \right) - \right. \\ & \left. - \operatorname{div} f(0) \right] t e^{-tg(0)} k(t; x - t f(0)). \end{aligned}$$

We thus see that the $k_\varepsilon(t; x, 0)$ has a smooth first derivative. Similar computations show the existence of all higher ε derivatives of $k_\varepsilon(t; x, 0)$, and thus show that $k_\varepsilon(t; x, 0)$ is smooth in ε at $\varepsilon = 0$. The same techniques extend to show smoothness in ε of $k_\varepsilon(t; x, y)$ at any ε and y . We also remark that the ε derivatives of $k_\varepsilon(t; 0, 0)$ at $\varepsilon = 0$ are all computed by expressions involving $f^i(0)$, $g(0)$ and $k(t; x)$. \square

Chapter 4

A local index theorem

This chapter proves a local index theorem for Dirac operators coupled to superconnections. As in much of the previous chapter, many of the arguments used are essentially found in [17], with care taken to take the superconnection scaling into account.

Theorem 4.1. *Let X be a compact, spin and Riemannian manifold. Let $E \rightarrow X$ be a complex, hermitian and finite dimensional \mathbb{Z}_2 -graded vector bundle, with ∇ a unitary superconnection on E . Let $p_{t,s}(x, y)$ be the heat kernel associated to $\mathcal{D}(\nabla^s)$. Then*

$$\lim_{t \rightarrow 0} \text{Tr } p_{t,1/t}(x, x) \, dx = (2\pi i)^{-n/2} \left[\hat{A}(\Omega^X) \text{ch } \nabla \right]_{(n)},$$

where $n = \dim X$.

Before proving the theorem, let us note that as a corollary we obtain the index theorem for superconnections. It will be convenient to introduce some notation. The Dirac operator $\mathcal{D}(\nabla)$ may be written

$$\mathcal{D}(\nabla) = \begin{pmatrix} & \not{D}(\nabla)^* \\ \not{D}(\nabla) & \end{pmatrix}$$

in terms of the grading on $\Gamma(\mathbb{S} \otimes E)$, where $\not{D}(\nabla) : \Gamma(\mathbb{S} \otimes E)^{\text{ev}} \rightarrow \Gamma(\mathbb{S} \otimes E)^{\text{odd}}$.

Corollary 4.2. *Let X be a compact, spin and Riemannian manifold. Let $E \rightarrow X$ be a complex, hermitian and finite dimensional \mathbb{Z}_2 -graded vector bundle with ∇ a unitary superconnection on E . Then*

$$\text{index } \not{D}(\nabla) = (2\pi i)^{-n/2} \int_X \hat{A}(\Omega^X) \text{ch } \nabla.$$

Proof. We need to argue that

$$\lim_{t \rightarrow 0} \int_X dx \text{Tr } p_{t,1/t}(x, x) = \text{index } \not{D}(\nabla).$$

We will prove the McKean-Singer formula [27]

$$\int_X dx \text{Tr } p_{t,1/t}(x, x) = \text{index } \not{D}(\nabla^{1/t}),$$

which gives the result, as the index is an invariant of the homotopy class of a Fredholm operator.

Let us fix $t \in \mathbb{R}^+$. For the remainder of the argument we will write $\not{D} = \not{D}(\nabla^{1/t})$. We may decompose $L^2(X; \mathbb{S} \otimes E)$ in terms of eigenvalues of $\mathcal{D}(\nabla^{1/t})^2$. The latter is an even operator, so the decomposition respects the grading: on $L^2(X; \mathbb{S} \otimes E)^{\text{ev}}$, $\not{D}^* \not{D}$ is diagonalised, and on $L^2(X; \mathbb{S} \otimes E)^{\text{odd}}$, $\not{D} \not{D}^*$ is. Suppose now that ϕ_λ is an eigenvector of $\not{D}^* \not{D}$, with eigenvalue $\lambda \neq 0$. Then $\not{D} \phi_\lambda$ is an eigenvector of $\not{D} \not{D}^*$ with eigenvalue λ . Similarly, every eigenvector of $\not{D} \not{D}^*$ with non-zero eigenvalue is mapped by \not{D}^* to an eigenvector of $\not{D}^* \not{D}$ with the same eigenvalue. We thus see that for every non-zero eigenvalue the eigenspaces of $\not{D}^* \not{D}$ and $\not{D} \not{D}^*$ are isomorphic, with \not{D} giving the isomorphism. The zero eigenspaces are *not* in general isomorphic. Indeed, the difference in

dimension between the kernel of $\not{D}^* \not{D}$ and of $\not{D} \not{D}^*$ is precisely the index of \not{D} .

The following lemma is an easy consequence of these considerations.

Lemma 4.3. *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $F(0) = 1$, and*

$$\sum_{\lambda \in \text{spec } \not{D}^* \not{D}} |F(\lambda)| < \infty.$$

Then

$$\begin{aligned} \text{index } \not{D} &= \sum_{\lambda \in \text{spec } \not{D}^* \not{D}} F(\lambda) - \sum_{\lambda \in \text{spec } \not{D} \not{D}^*} F(\lambda) \\ &= \text{Tr } F(\mathcal{D}(\nabla^{1/t})^2). \end{aligned}$$

We saw in the proof of Prop. 3.16 that the function $e^{-t\lambda}$ satisfies the hypotheses of the lemma. We thus have that

$$\text{index } \not{D} = \text{Tr } e^{-t\mathcal{D}(\nabla^{1/t})^2}.$$

We need to see that

$$\text{Tr}_0 e^{-t\not{D}^* \not{D}} = \int_X dx p_{t,1/t}^{\text{ev}}(x, x),$$

converges, where $p_{t,1/t}^{\text{ev}}(x, y)$ is the heat kernel for $\not{D}^*(\nabla^{1/t})\not{D}(\nabla^{1/t})$, and Tr_0 is the ungraded trace. But the heat kernel is smooth, and by the compactness of X the integral converges. \square

4.1 Getzler's scaling

In order to prove Th. 4.1 we will combine the algebraic scaling in 2.4 and geometric scaling used in proving 3.36. We now fix our compact, spin

and Riemannian manifold X and complex, hermitian and finite dimensional \mathbb{Z}_2 -graded vector bundle $E \rightarrow X$ with superconnection ∇ ; and form the Dirac operator $\mathcal{D}(\nabla)$. We also fix a decomposition $\nabla^s = \nabla + \omega^s$, where ∇ is an ordinary connection, and we recall

$$\nabla^s = \sum_i |s|^{(1-i)/2} [\nabla]_{(i)}.$$

Let us focus our attention near a point $x \in X$. As in Ch. 3 we choose an $r \in \mathbb{R}^+$ smaller than the injectivity radius of X , and identify $B_r \subset T_x(X)$ and an open neighbourhood $x \in U \subset X$ using the exponential map. We use the map to transfer the geometry near x to $T_x(X)$, and extend it as before. We thus have \mathbb{Z}_2 -graded, complex and hermitian vector bundles $\mathbb{S} \rightarrow T_x(X)$ with the Levi-Civita connection $\nabla^{\mathbb{S}}$, $E \rightarrow T_x(X)$ with superconnection ∇ , and metric g , all of which near the origin are pulled back from X via the exponential map. We recall there is a scaling map $T_\varepsilon : T_x(X) \rightarrow T_x(X)$ given by $T_\varepsilon(\xi) = \varepsilon\xi$ for $\xi \in T_x(X)$. We combine the algebraic scaling in 2.4 and the geometric scaling to give the scaling operator

$$S_\varepsilon = U_\varepsilon T_\varepsilon^*.$$

We define a family of heat operators

$$\begin{aligned} \tilde{H}_\varepsilon &\equiv \frac{\partial}{\partial t} + P_\varepsilon, \\ P_\varepsilon &\equiv \varepsilon^2 S_\varepsilon \mathcal{D}(\nabla^{s/\varepsilon^2}) S_\varepsilon^{-1}. \end{aligned} \tag{4.4}$$

The following lemma is crucial.

Lemma 4.5. *Let $\{\xi^k\}$ be an orthogonal linear co-ordinate system on $T_x X$.*

Then $\lim_{\varepsilon \rightarrow 0} P_\varepsilon$ exists, and is given by

$$\lim_{\varepsilon \rightarrow 0} P_\varepsilon = P_0 = - \sum_{k,l} \left(\frac{\partial}{\partial \xi^k} - \frac{1}{4} e(\Omega_{kl}^X(x)) \xi^l \right)^2 + e((\nabla^s)^2)(x).$$

Proof. The Weitzenböck formula 3.1 asserts

$$\begin{aligned} \mathcal{D}(\nabla^s)^2 &= \nabla^* \nabla + \frac{R}{4} + Q^s, \\ Q^s &= c(\Omega^E) + [c(\nabla^E), c(\omega^s)] + c(\omega^s)^2, \end{aligned}$$

where $\nabla^s = \nabla^E + \omega^s$, $\nabla^* \nabla$ is the covariant laplacian constructed from $\nabla^{\mathbb{S}} \otimes 1 + 1 \otimes \nabla^E$, and $\Omega^E = \text{Curv}(\nabla^E)$. We examine each term in turn.

It is a standard result that (see, for instance, [10] Ch. 4)

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 S_\varepsilon \nabla^* \nabla S_\varepsilon^{-1} = - \sum_{k,l} \left(\frac{\partial}{\partial \xi^k} - \frac{1}{4} e(\Omega_{kl}^X(x)) \xi^l \right)^2.$$

The scalar curvature is easily seen to vanish as $\varepsilon \rightarrow 0$. We are thus left with $\lim_{\varepsilon \rightarrow 0} \varepsilon^2 S_\varepsilon Q^{s/\varepsilon^2} S_\varepsilon^{-1}$. Applying lemma 2.13 shows

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 S_\varepsilon Q^{s/\varepsilon^2} S_\varepsilon^{-1} = e((\nabla^s)^2)(x)$$

as required. □

We now study the heat operator

$$\tilde{H}_0 = \frac{\partial}{\partial t} + P_0.$$

Proposition 4.6. *The heat kernel for \tilde{H}_0 at the origin is given by the formula*

$$\tilde{p}_{t,u}(0,0) = (4\pi t)^{-n/2} \sqrt{\det \left(\frac{t\Omega^X/2}{\sinh(t\Omega^X/2)} \right)} e^{-t(\nabla^s)^2},$$

where all curvatures are evaluated at the origin.

Proof. For simplicity we write

$$P_0 = L + F,$$

where

$$L = - \sum_{k,l} \left(\frac{\partial}{\partial \xi^k} - \frac{1}{4} e(\Omega_{kl}^X(x)) \xi^l \right)^2,$$

and

$$F = e((\nabla^s)^2).$$

We note that L and F commute. Thus the heat semigroup associated to P_0 factorises as

$$e^{-tP_0} = e^{-tL} e^{-tF}.$$

We thus need to analyse the kernel of e^{-tL} . Now, the curvature $\Omega^X(0)$ is a totally skew matrix with values in two-forms, and co-ordinates may thus be chosen such that

$$\Omega^X = \begin{pmatrix} 0 & -\omega_1 & & & \\ \omega_1 & 0 & & & \\ & & 0 & -\omega_2 & \\ & & \omega_2 & 0 & \\ & & & & \ddots \end{pmatrix}.$$

The operator L thus decomposes as a sum of commuting operators of the form

$$\begin{aligned} & - \left(\frac{\partial}{\partial x} + \frac{1}{4}\omega y \right)^2 - \left(\frac{\partial}{\partial y} - \frac{1}{4}\omega x \right)^2 \\ & = - \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) - \frac{1}{16}\omega^2 (x^2 + y^2) - \frac{1}{2}\omega \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right). \end{aligned}$$

Examining the right hand side, we notice that the first two terms are rotationally invariant, whereas the last annihilates rotationally invariant functions.

We may thus drop the last term, and find the kernel for

$$\frac{\partial}{\partial t} - \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) - \frac{1}{16}\omega^2 (x^2 + y^2).$$

We note that this is a sum of equations for x and y : the kernels for these are given by Mehler's formula (Eq. 4.13 in Sec. 4.2), and thus the kernel at the origin for e^{-tP_0} is

$$(4\pi t)^{-n/2} \sqrt{\det \left(\frac{t\Omega^X/2}{\sinh(t\Omega^X/2)} \right)}.$$

□

Corollary 4.7. *The heat kernel $\tilde{p}_{t,u}^0(x, y)$ has the small time asymptotic expansion*

$$\tilde{p}_{t,u}^0(0, 0) = (4\pi t)^{-n/2} \sum_{j \geq 0} P_j(\Omega^X/2, -(\nabla^s)^2) t^j$$

at the origin, where $P_j(x, y)$ is a homogeneous polynomial of degree j .

We now return to analyse the family of heat operators \tilde{H}_ε in Eq. 4.4.

We define

$$\tilde{p}_{t,s}^\varepsilon(x, y) = \varepsilon^n S_\varepsilon p_{\varepsilon^2 t, s/\varepsilon^2}(x, y). \quad (4.8)$$

A short computation shows that $\tilde{H}_\varepsilon \tilde{p}_{t,s}^\varepsilon(x, y) = 0$, with initial condition δ_x , so that, by uniqueness of solutions, $\tilde{p}_{t,s}^\varepsilon(x, y)$ is the heat kernel for \tilde{H}_ε . Fixing an orthonormal basis $\{e^i\}$ of $T_x^*(X)$, we write

$$p_{t,s}(x, y) = \sum_I p_{t,s}^I(x, y) e^I,$$

where the I are ordered multi-indices, the $e^I \in \text{Cl}(X)_x$, and $p_{s,t}^I(x, y) \in \text{End}(E_x)$. Then

$$\tilde{p}_{t,s}^\varepsilon(x, y) = \sum_I \varepsilon^{n-|I|} p_{\varepsilon^2 t, s/\varepsilon^2}^I(x, \varepsilon y) \hat{e}^I.$$

The small t asymptotic expansion of $p_{t,s/t}(x, x)$ derived in Ch. 3, prop. 3.36, may be written

$$p_{t,s/t}(x, x) \sim (4\pi)^{-n/2} \sum_{j,I} t^j e^I A_{j,I},$$

so that

$$\tilde{p}_{t,s/t}^\varepsilon(x, x) \sim (4\pi)^{-n/2} \sum_{j,I} \varepsilon^{2j-|I|} t^{j-n/2} \hat{e}^I A_{j,I}.$$

It is well known (see [10] Ch. 3) that

$$\text{Tr } \hat{e}^I = \begin{cases} (-2i)^{n/2} & \text{if } I \sim 12 \dots n, \\ 0 & \text{otherwise,} \end{cases}$$

and we compute

$$\text{Tr } \tilde{p}_{t,s}^\varepsilon(x, x) \sim (4\pi)^{-n/2} \sum_j \varepsilon^{2j-n} t^{j-n/2} \text{Tr } A_{j,12\dots n}.$$

Cor. 4.7 shows that

$$\tilde{p}_{t,s/t}^0(x, x) = (4\pi)^{n/2} \sum_j P_j(\Omega^X/2, -(\nabla^{s/t})^2) t^{j-n/2}.$$

The coefficient of $t^{j-n/2}$ is a $2j$ -form. On the other hand, the coefficient of $t^{j-n/2}$ is given by

$$\lim_{t \rightarrow 0} \sum_I \varepsilon^{2j-|I|} \hat{e}^I A_{j,I}.$$

Thus $A_{j,I} = 0$ for $2j < |I|$. We see then that

$$\lim_{t \rightarrow 0} \text{Tr } p_{t,s/t}(x, x) = \text{Tr } A_{n/2, 12 \dots n}.$$

However $A_{n/2, 12 \dots n}$ is the constant term in the small t asymptotic expansion of $\tilde{p}_{t,s/t}^0(x, x)$. Examining this asymptotic expansion, and noting that the k -form part of $t(\nabla^{u/t})^2$ is order $t^{k/2}$, we see that

$$\text{Tr } A_{n/2, 12 \dots n} = \lim_{t \rightarrow 0} \text{Tr } \tilde{p}_{t,s/t}(x, x) = (2\pi i)^{n/2} \left[\hat{A}(\Omega^X) \text{ch}(\nabla^s) \right]_{(n)},$$

proving the result.

Remark 4.9. We note the essential role the \mathbb{R}^\times -action on superconnections played in the proof.

4.2 Mehler's formula

Consider the equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + a^2 x^2 u = 0, \tag{4.10}$$

where $u(x, t)$ is a time varying function on \mathbb{R} . We wish to solve this subject to the initial condition

$$\lim_{t \rightarrow 0} u(\cdot, t) = \delta_0.$$

We make the Ansatz

$$u(x, t) = A(t)e^{-\mu(t)x^2/2}.$$

Substituting this into Eq. 4.10 we obtain two ordinary differential equations:

$$\frac{d(\log A)}{dt} = -\mu, \quad (4.11)$$

$$\frac{d\mu}{dt} = 2(a^2 - \mu^2). \quad (4.12)$$

Eq. 4.12 is solved by

$$\mu(t) = a \coth(2at + C),$$

where C is to be determined. Substituting this into 4.11 allows us to conclude

$$A(t) = \sqrt{\frac{C'}{\sinh(2at + C)}},$$

where C' is undetermined. Thus

$$u(x, t) = \sqrt{\frac{C'}{\sinh(2at + C)}} \exp \left[-\frac{ax^2}{2 \tanh(2at + c)} \right].$$

Comparing with the initial condition allows us to determine C and C' , finally giving the Mehler's formula for the solution:

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \left(\frac{2at}{\sinh 2at} \right)^{1/2} \exp \left[\frac{-x^2}{4t} \left(\frac{2at}{2 \tanh 2at} \right) \right]. \quad (4.13)$$

Chapter 5

Manifolds with boundary

The goal of this chapter is to define η -invariants for superconnections, and show how they give rise to an index theorem for manifolds with boundary. We shall begin by reviewing the pertinent theory for η -invariants associated to connections.

5.1 The η -invariant associated to a connection

Let Y be an odd-dimensional compact and spin Riemannian manifold, and $V \rightarrow Y$ a complex, hermitian and finite dimensional vector bundle with unitary connection ∇ . Let $n = \dim Y$. We recall that $\omega_n \equiv i^{n(n+1)/2} c(\text{vol})$ is an odd endomorphism of the bundle of spinors $\mathbb{S}(Y) \rightarrow Y$ that squares to the identity and commutes with the Dirac operator $\mathcal{D}(\nabla)$. The operator $\omega \mathcal{D}(\nabla)$ is even and self-adjoint, and $(\omega \mathcal{D}(\nabla))^2 = \mathcal{D}(\nabla)^2$. Diagonalising the laplacian associated with $\mathcal{D}(\nabla)$ thus diagonalises $\omega \mathcal{D}(\nabla)$, and one sees that its spectrum is real and discrete. Morally speaking, the η -invariant computes the discrepancy between its positive and negative eigenvalues:

$$\eta_Y(\nabla) = \sum_{\lambda \in \text{spec } \omega \mathcal{D}(\nabla)} \text{sign } \lambda, \quad (5.1)$$

where we recall the decomposition

$$\mathcal{D}(\nabla) = \begin{pmatrix} & \not{D}(\nabla) \\ \not{D}(\nabla) & \end{pmatrix}.$$

In order to make sense of eq. 5.1, one defines the function

$$\eta_Y(\nabla)(s) = \sum_{\lambda \in \text{spec } \omega \not{D}(\nabla)} \frac{\lambda}{|\lambda|^{1+s}}. \quad (5.2)$$

For $\text{Re}(s)$ large enough $\eta_Y(\nabla)(s)$ is holomorphic, and one may show that there is a unique analytic continuation to $s = 0$ [3, 5]. One defines the η -invariant of ∇ to be $\eta_Y(\nabla)(0)$ [3]. There are two auxiliary quantities related to the η -invariant. The first is the so-called *reduced* η -invariant, $\xi_Y(\nabla)$, defined by

$$\xi_Y(\nabla) = \frac{\eta_Y(\nabla) + \ker \not{D}(\nabla)}{2}, \quad (5.3)$$

and the second is the τ -invariant, given by

$$\tau_Y(\nabla) = \exp(2\pi i \xi_Y(\nabla)). \quad (5.4)$$

Suppose now that X is an even-dimensional compact and spin Riemannian manifold with boundary Y , and that $V \rightarrow Y$ extends to a complex and hermitian vector bundle $V \rightarrow X$ with unitary connection ∇ . Suppose further that there exists an open neighbourhood $\partial X \subset U$ diffeomorphic to $Y \times [0, 1)$ such that the geometry on U is pulled back from the geometry on Y extended trivially to $Y \times [0, 1)$. Define the operator $P : L^2(\mathbb{S}(Y) \otimes V) \rightarrow L^2(\mathbb{S}(Y) \otimes V)$ as the orthogonal projection to the span of the eigensections of $\not{D}(\nabla)$ with positive eigenvalue. The Atiyah-Patodi-Singer boundary conditions are defined

in terms of P : the Dirac operator $\mathcal{D}(\nabla)$ on X is restricted to act on sections that are annihilated by P on the boundary. Restricting to such sections makes $\mathcal{D}(\nabla)$ elliptic.

The fundamental theorem for η -invariants is the APS theorem [3], which computes the index of the Dirac operator on X under APS boundary conditions in terms of the η -invariant.

Theorem 5.5. *Let X be an even dimensional, compact and spin Riemannian manifold with $\partial X = Y$. Let $V \rightarrow X$ be a finite dimensional, complex and hermitian vector bundle, with unitary connection ∇ . Suppose further that there is some neighbourhood $U \subset X$ of Y , with $U \cong Y \times [0, 1)$, such that the data restricted to U is pulled back from that on Y extended trivially to $Y \times [0, 1)$. Then*

$$\text{index } \mathcal{D}(\nabla) = (2\pi i)^{-\dim X/2} \int_X \hat{A}(\Omega^X) \text{ch}(\nabla) + \xi_Y(\nabla),$$

where the index is computed for the Dirac operator on sections $\psi \in \mathbb{S}(X) \otimes V$ satisfying $P\psi|_Y = 0$.

5.2 Superconnections

We would like to generalise the preceding theory to superconnections. Unfortunately the usual definition used for the η -invariant does not yield a satisfactory definition in this context: the index theorem uses an entire family of Dirac operators $\mathcal{D}(\nabla^t)$ which cannot simultaneously be diagonalised, making a definition in terms of eigenvalues difficult. We will instead approach

the definition of an appropriate η -invariant from a geometric viewpoint, and use the APS-theorem as a starting point. In the end we will only define a τ -invariant for superconnections.

Before discussing superconnections, let us pause to consider the η -invariant for an ordinary connection on a \mathbb{Z}_2 -graded vector bundle. Let $V \rightarrow Y$ be a complex and hermitian \mathbb{Z}_2 -graded vector bundle with unitary connection ∇ over an odd-dimensional compact and spin Riemannian manifold. One may define the η -invariant associated with ∇ just as for an ungraded vector bundle, and when one takes signs into account one sees that

$$\eta_Y(\nabla) = \eta_Y(\nabla^{\text{ev}}) - \eta_Y(\nabla^{\text{odd}}),$$

where $\nabla^{\text{ev}}, \nabla^{\text{odd}}$ are ∇ restricted to $V^{\text{ev}}, V^{\text{odd}}$ respectively, seen as ungraded vector bundles.

In order to proceed, we make a useful definition.

Definition 5.6. Let Y be a smooth manifold, and $V \rightarrow Y$ be a finite dimensional, \mathbb{Z}_2 -graded vector bundle, with superconnections ∇_0, ∇_1 . The *Chern-Simons form* associated with the pair of superconnections is given by

$$\alpha[\nabla_0, \nabla_1] = \int_0^1 dt \operatorname{Tr} \left[(\nabla_1 - \nabla_0) e^{((1-t)\nabla_0 + t\nabla_1)^2} \right].$$

We may now define the τ -invariant associated to a superconnection.

Definition 5.7. Let Y be an odd-dimensional, compact and spin Riemannian manifold, and $V \rightarrow Y$ be a finite dimensional, complex and hermitian

\mathbb{Z}_2 -graded vector bundle with unitary superconnection ∇ . The τ -invariant associated to this data is given by

$$\tau_Y(\nabla) = \tau_Y(\nabla) \exp \left[(2\pi i)^{(1-\dim Y)/2} \int_Y \hat{A}(\Omega^Y) \alpha[\nabla, \nabla] \right],$$

where we have written $\nabla = \nabla + \omega$, and ∇ is a connection on V .

We will see in the next section that this is well-defined.

Motivation. The motivation for the definition comes from a particular application of the APS theorem. Let $Y, V \rightarrow Y$, $\nabla = \nabla + \omega$ be as in the definition. Form $\tilde{Y} = Y \times [0, 1]$, $\tilde{V} = V \times [0, 1]$. Define the monotone smooth function $\rho \in C^\infty(\mathbb{R})$ such that

$$\begin{cases} \rho(t) = 0 & t \leq 0.1, \\ \rho(t) = 1 & 0.9 < t, \end{cases} \quad (5.8)$$

and let $\tilde{\nabla} = \rho(t)\nabla + (1 - \rho(t))\nabla$. If the APS-theorem were to hold for τ -invariants of superconnections, we would have

$$\tau_Y(\nabla) = \tau_Y(\nabla) \exp \left[(2\pi i)^{(1-\dim Y)/2} \int_{\tilde{Y}} \hat{A}(\Omega^{\tilde{Y}}) \text{ch } \tilde{\nabla} \right].$$

It is an easy exercise to see that

$$\int_{\tilde{Y}} \hat{A}(\Omega^{\tilde{Y}}) \text{ch } \tilde{\nabla} = \int_Y \hat{A}(\Omega^Y) \alpha[\nabla, \nabla].$$

□

5.3 An APS theorem

The τ -invariant defined in the previous section obeys a generalisation of the APS theorem.

Theorem 5.9. *Let X be a compact and spin Riemannian manifold with boundary $\partial X = Y$. Let $E \rightarrow X$ be a finite dimensional, complex, hermitian, \mathbb{Z}_2 -graded vector bundle, and ∇ a unitary superconnection on E . Suppose there exists an open neighbourhood $Y \subset U \subset X$ homeomorphic to $Y \times [0, 1]$ such that the geometry on U is pulled back from that on Y extended trivially. Then*

$$\tau_Y(\nabla|_Y) = \exp \left[-(2\pi i)^{1-\dim X/2} \int_X \hat{A}(\Omega^X) \text{ch}(\nabla) \right].$$

Before proving the theorem, let us see how it shows the τ -invariant to be well-defined.

Corollary 5.10. *The τ -invariant defined in 5.7 is well-defined.*

Proof. The only choice in def. 5.7 is the choice of connection in the decomposition $\nabla = \nabla + \omega$. Let us form the $\tau'_Y(\nabla)$ with a different decomposition $\nabla = \nabla' + \omega'$. Form $\tilde{Y} = Y \times [0, 1]$, $\tilde{E} = E \times [0, 1]$, with superconnection $\tilde{\nabla} = \rho(t)\nabla + (1 - \rho(t))\nabla'$, where $\rho(t)$ is as in Eq. 5.8. The APS-theorem applied to this data then shows

$$\begin{aligned} \tau'_Y(\nabla) &= \tau_Y(\nabla) \exp \left[(2\pi i)^{(1-\dim Y)/2} \int_{\tilde{Y}} \hat{A}(\Omega^Y) \text{ch}(\tilde{\nabla}) \right] \\ &= \tau_Y(\nabla) \exp \left[(2\pi i)^{(1-\dim Y)/2} \int_Y \hat{A}(\Omega^Y) \alpha[\nabla, \nabla'] \right] \\ &= \tau_Y(\nabla). \end{aligned}$$

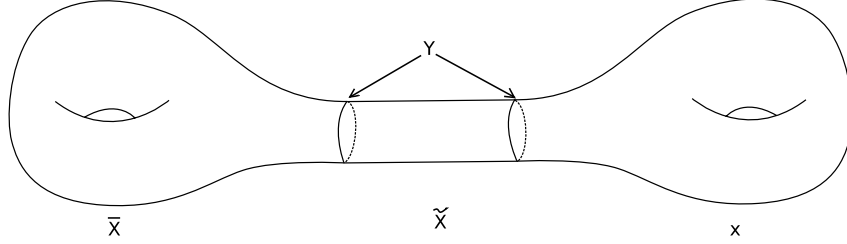


Figure 5.1: The manifold \mathcal{X} .

□

Proof of Theorem 5.9. Write $\nabla = \nabla + \omega$. Let $\bar{E} \rightarrow \bar{X}$, with $\bar{Y} = \partial\bar{X}$ denote the data with the opposite choice of orientation. Let $\tilde{Y} = Y \times_I$, $\tilde{E} = E|_Y \times I$, and let $\tilde{\nabla} = (1 - \rho(t))\nabla|_Y + \rho(t)\nabla|_Y$. Form

$$\mathcal{X} = X \cup_Y \tilde{X} \cup_Y \bar{X},$$

with $\mathcal{E} \rightarrow \mathcal{X}$ defined by

$$\mathcal{E} = E \cup_{E|_Y} \tilde{E} \cup_{E|_Y} \bar{E},$$

with superconnection $\nabla^{\mathcal{E}}$ glued from ∇ on E , $\tilde{\nabla}$ on \tilde{E} , and ∇ on \bar{E} . The situation is illustrated in figure 5.1. The index theorem applied to this situation shows

$$(2\pi i)^{-n/2} \int_{\mathcal{X}} \hat{A}(\Omega^{\mathcal{X}}) \text{ch}(\nabla^{\mathcal{E}}) = 0 \pmod{1}.$$

But

$$\begin{aligned} \int_{\mathcal{X}} \hat{A}(\Omega^{\mathcal{X}}) \text{ch} \nabla^{\mathcal{E}} = \\ \int_X \hat{A}(\Omega^X) \text{ch} \nabla + \int_Y \hat{A}(\Omega^Y) \alpha[\nabla|_Y, \nabla] + \int_{\bar{X}} \hat{A}(\Omega^{\bar{X}}) \text{ch} \nabla. \end{aligned}$$

Applying the APS theorem for ξ -invariants of connections, we have

$$0 = (2\pi i)^{-n/2} \int_X \hat{A}(\Omega^X) \operatorname{ch} \nabla + (2\pi i)^{-n/2} \int_Y \hat{A}(\Omega^Y) \alpha[\nabla|_Y, \nabla] - \xi_{\bar{X}}(\nabla)$$

modulo integers. Exponentiating both sides, and noting $\xi_{\bar{X}}(\nabla) = -\xi_X(\nabla)$, one obtains the theorem.

□

Chapter 6

Families of Dirac operators

6.1 Riemannian maps

The good notion of “family” for Riemannian geometry is that of a Riemannian map.¹

Definition 6.1 (Riemannian maps). Let X, Y be smooth manifolds. A *Riemannian map* is a triple $(\pi, g^{X/Y}, P)$, where $\pi : X \rightarrow Y$ is a smooth submersion, $g^{X/Y}$ be a positive, non-degenerate, symmetric bi-linear form on $T(X/Y)$, and $P : T(X) \rightarrow T(X/Y)$ a smooth projection.

Remark 6.2. Several remarks are in order. Generally, one simply denotes the Riemannian map by $\pi : X \rightarrow Y$, supressing the metric and projection. Specifying the projection P is equivalent to choosing a “horizontal” vector field in $T(X)$, and determines a splitting

$$T(X) \cong T(X/Y) \oplus \pi^*T(Y),$$

and in particular, allows one to take a horizontal lift of vector fields $\xi \in T(Y)$, which we denote by $\tilde{\xi}$. A choice of metric g^Y on $T(Y)$ makes X a Riemannian

¹The notation and presentation used to describe the geometry of families of Dirac operators is that in [18].

manifold, with metric

$$g^X = g^{X/Y} \oplus \pi^* g^Y. \quad (6.3)$$

Conversely, a Riemannian map $\pi : X \rightarrow Y$ along with a Riemannian structure on X such that the metric splits as in 6.3 is called a Riemannian *submersion*.

A Riemannian map $X \rightarrow \{\text{pt}\}$ is a Riemannian structure on X .

Proposition 6.4 (Levi-Civita connection). *Let $\pi : X \rightarrow Y$ be a Riemannian map. There exists a unique torsion free connection ∇^π on $T(X/Y)$ preserving $g^{X/Y}$. It is called the Levi-Civita connection of the map.*

Construction. Choose a Riemannian structure on Y and give X a Riemannian structure by setting $g^X = g^{X/Y} \oplus \pi^* g^Y$. The Levi-Civita theorem asserts that there exists a unique unitary torsionfree connection ∇^X on X preserving g^X . Form

$$\nabla^\pi \equiv P \nabla^X P.$$

One may check that this is independent of choices². □

There are two important curvatures associated with a Riemannian map $\pi : X \rightarrow Y$ in addition to the curvature Ω^π of ∇^π .

Definition 6.5. Let $\pi : X \rightarrow Y$ be a Riemannian map. There exist canonical tensors $\Pi^\pi : T(X/Y) \otimes T(X/Y) \otimes \ker P \rightarrow \mathbb{R}$ (the *second fundamental form*) and $T^\pi : T(Y) \otimes T(Y) \rightarrow T(X/Y)$ depending only on the Riemannian map.

²Details of all constructions in this section may be found in [10] Ch. 10.

The tensor Π^π is symmetric in the first two factors; T^π is totally skew, and does not depend on $g^{X/Y}$.

Construction. To define Π^π , we choose a Riemannian structure g^Y on Y , and induce a Riemannian structure on X . Define

$$\Pi^\pi : \alpha \otimes \beta \otimes \xi \mapsto -g^{X/Y}(P\nabla_\alpha^X \xi, \beta),$$

where $\alpha, \beta \in T(X/Y)$, $\xi \in \ker P$. One may show that this is independent of the choice of Riemannian structure on Y .

We define T^π as follows. Let $\xi, \chi \in T(Y)$. Then

$$T^\pi : \xi, \chi \mapsto P([\tilde{\xi}, \tilde{\chi}]).$$

□

We may define the mean curvature of the family from the second fundamental form.

Definition 6.6. The mean curvature $H^\pi \in \Omega^1(X)$ is a horizontal one-form defined as follows. Let $\xi \in T(X)$, and decompose ξ as a sum of horizontal and vertical vectors:

$$\xi = P\xi + \zeta.$$

Let $\{e_i\}$ be an orthonormal basis for $T(X/Y)$. Then

$$H^\pi(\xi) = \text{Tr } \Pi^\pi(\cdot, \cdot, \zeta) = \sum_i \Pi^\pi(e_i, e_i, \zeta).$$

There is a natural \mathbb{R}^\times -action associated with a Riemannian map $\pi : X \rightarrow Y$ which scales the vertical metric. It is given by

$$\pi^t = (\pi, |t|^{-1} g^{X/Y}, P). \quad (6.7)$$

In order to understand the effect of the action, let us introduce a second Riemannian map $\rho : Y \rightarrow \{\text{pt}\}$ (in other words, a Riemannian structure on Y). As remarked earlier, this gives X a Riemannian structure, with metric $g^X = g^{X/Y} \oplus \pi^* g^Y$. The Levi-Civita connection associated to this metric is $\nabla^{\rho \circ \pi}$. The Levi-Civita connection is independent of a global scaling of the metric, and thus

$$\nabla^{\rho^t \circ \pi} = \nabla^{\rho \circ \pi^{1/t}}. \quad (6.8)$$

The limit $t \rightarrow 0$ of $\rho^t \circ \pi$ is important, and is called the *adiabatic limit*: for t very small the geometry of the base may be thought to be changing very slowly (adiabatically) compared to that on the fiber. We will sometimes denote the adiabatic limit of a geometric quantity Φ by $\text{a-lim } \Phi$. The following proposition summarises the geometry of the adiabatic limit.

Proposition 6.9. *The limit $\text{a-lim } \nabla^X = \lim_{t \rightarrow 0} \nabla^{\rho^t \circ \pi}$ exists, and is torsion free. It is incompatible with any Riemannian structure on X . Relative to the decomposition $T(X) \cong T(X/Y) \oplus T(B)$, the connection may be written*

$$\text{a-lim } \nabla^X = \begin{pmatrix} \nabla^\pi & * \\ & \nabla^\rho \end{pmatrix},$$

and thus

$$\text{a-lim } \Omega^X = \begin{pmatrix} \Omega^\pi & * \\ & \Omega^\rho \end{pmatrix}.$$

In particular, in the adiabatic limit the form $\hat{A}(\Omega^X)$ exists, and decomposes as

$$\text{a-lim } \hat{A}(\Omega^X) = \pi^* \hat{A}(\Omega^Y) \wedge \hat{A}(\Omega^\pi).$$

The proposition is proved in [10] Ch. 10.

Remark 6.10. The Clifford bundle $\text{Cl}(X)$ also has an adiabatic limit:

$$\text{a-lim } \text{Cl}(X) = \pi^* \wedge T^*(Y) \otimes \text{Cl}(X/Y),$$

with the horizontal multiplication degenerating to exterior multiplication. If the fibres of π are spin, then the bundle $\mathbb{S}_0 = \pi^* \Omega(Y) \otimes \mathbb{S}(X/Y)$ is a left module for $\text{a-lim } \text{Cl}(X)$, with the action being Clifford multiplication in vertical directions, and exterior multiplication in horizontal directions. We denote this action by $c_0(\cdot)$. The module \mathbb{S}_0 plays a similar role to spinors.

We recall the discussion in Ex. 2.4 that the Clifford algebra of a vector space with metric is naturally isomorphic as a vector space to the exterior algebra. Thus there is an isomorphism of vector bundles

$$\sigma : \text{Cl}(X/Y) \rightarrow \bigwedge T^*(X/Y), \quad (6.11)$$

which we call the symbol map. We extend this to an isomorphism of vector bundles

$$\text{a-lim } \text{Cl}(X) \cong \pi^* \bigwedge T^*(Y) \otimes \text{Cl}(X/Y) \rightarrow \bigwedge T^*(X). \quad (6.12)$$

There is a natural bigrading on $\bigwedge T^*(X)$ given by

$$\bigwedge T^*(X) = \sum_{p,q} \bigwedge^{p,q} T^* X, \quad (6.13)$$

$$\bigwedge^{p,q} T^* X = \bigwedge^p T^*(Y) \otimes \bigwedge^q T^*(X). \quad (6.14)$$

We denote the (p, q) component $\alpha \in \Omega(X)$ by $[\alpha]_{(p,q)}$.

6.2 Pushforwards

Let $\pi : X \rightarrow Y$ be a Riemannian map, with compact and spin fibres, and $E \rightarrow X$ a complex, hermitian, finite dimensional, \mathbb{Z}_2 -graded vector bundle with unitary superconnection ∇ . We wish to construct an infinite dimensional complex and hermitian \mathbb{Z}_2 -graded vector bundle $\pi_! E \rightarrow Y$ with superconnection $\pi_! \nabla$ that represents the pushforward of this data to Y . This is to be done in a functorial manner, and it is to be done in such a way that, on choosing a metric on Y , $\mathcal{D}_Y(\pi_! \nabla) = \mathcal{D}_X(\nabla)$. This construction will extend the construction of Bismut [11] to superconnections.

We begin by examining the simplest case. Let $\mathbf{1} \rightarrow X$ be the purely even trivial line bundle with (super)connection d . We define $\pi_! \mathbf{1}$ to be the \mathbb{Z}_2 -graded Hilbert space bundle with typical fibre $\Gamma(\mathbb{S}(X/Y)_y)$, where $y \in Y$, and $\mathbb{S}(X/Y) \rightarrow X$ denotes the bundle of vertical spinors. The bundle $\pi_* \mathbf{1}$ has a different character depending on whether we take sections to be smooth or merely L^2 : in the former case it is a smooth Hilbert space bundle, in the latter it is continuous. We will take the sections to be smooth as we wish to define superconnections, but occasionally in doing analysis we will have to consider the bundle with fibres the L^2 sections. The bundle $\pi_! \mathbf{1} \rightarrow Y$ is hermitian, the metric being the L^2 -metric on the fibres.

The bundle $\pi_* \mathbf{1}$ possesses a natural unitary connection $\nabla^{\pi_! \mathbf{1}}$. In order to describe it, we begin by noting that the horizontal differential of the relative

volume form is just given by

$$\nabla_{\xi}^{\pi} \text{vol}^{X/Y} = -H^{\pi}(\xi) \text{vol}^{X/Y},$$

where ξ is a horizontal vector field. The appropriate connection on $\pi_! \mathbf{1}$ is thus given by

$$\nabla_{\xi}^{\pi_! \mathbf{1}} = \nabla_{\pi^* \xi}^{\pi} - \frac{1}{2} H^{\pi}(\pi^* \xi),$$

for $\xi \in T(Y)$.

We must introduce one more notion before we are in a position to define the pushforward of superconnections. Let $E \rightarrow X$ be a finite dimensional vector bundle with superconnection ∇ . Then the fibrewise Dirac operator $\mathcal{D}^{\pi}(\nabla) \in \Omega^0(Y; \text{End}(\pi_! \mathbf{1})^{\text{odd}})$ is defined to be the Dirac operator of ∇ restricted to the fibre: over a typical point $y \in Y$, $\mathcal{D}^{\pi}(\nabla)_y = \mathcal{D}(\nabla|_{\pi^{-1}y})$.

With this understood, we may now examine pushforwards of superconnections. The trivial bundle $\mathbf{1} \rightarrow X$ has a canonical superconnection, the de Rham differential d . Following Bismut, we will define the pushforward of d to be a unitary superconnection $\pi_! d$ on $\pi_! \mathbf{1}$. We will be able to define pushforwards of a general superconnection in terms of this.

Definition 6.15. Let $\pi : X \rightarrow Y$ be a Riemannian family with compact and spin fibres, and $\mathbf{1} \rightarrow X$ be the even and trivial complex \mathbb{Z}_2 -graded line bundle, with superconnection the de Rham differential d . Let $\pi_! \mathbf{1} \rightarrow Y$ be as above. Then the pushforward of the de Rham differential, $\pi_! d$ on $\pi_! \mathbf{1}$ is a superconnection given by

$$\pi_! d = \mathcal{D}^{\pi}(d) + \nabla^{\pi_! \mathbf{1}} + \frac{1}{4} c^{\pi}(T^{\pi}),$$

where c^π is vertical Clifford multiplication.

Remark 6.16. Several remarks are in order. First, we note that

$$\mathcal{D}_Y(\pi_! d) = \mathcal{D}_X(d).$$

Second, we note that if $Y = \{\text{pt}\}$, we have that $\pi_! \mathbf{1} \rightarrow \{\text{pt}\}$ is simply the bundle of spinors on X , seen as a \mathbb{Z}_2 -graded vector space, and the pushforward $\pi_! d$ is exactly the standard Dirac operator on X . On the other extreme, if we take $X = Y$, so that fibres are single points, then $\pi_! \mathbf{1} \rightarrow Y$ is just the trivial even line bundle on Y , and $\pi_! d$ is the de Rham differential on the bundle.

We now wish to extend our discussion to a general \mathbb{Z}_2 -graded vector bundle over a Riemannian family. We need one more definition.

Definition 6.17. Let $\pi : X \rightarrow Y$ be a Riemannian family, with compact and spin fibres. Let $\omega \in \Omega^i(X)$. Then the pushforward $\pi_! \omega \in \Omega(Y; \text{End}(\pi_! \mathbf{1}))$ is the sum

$$\pi_! \omega = \sum_{j \leq i} [\pi_! \omega]_{(j)},$$

where $[\pi_! \omega]_{(j)} \in \Omega^j(Y; \text{End}(\pi_! \mathbf{1}))$ is defined as

$$[\pi_! \omega]_{(j)}(\tilde{\xi}^1, \dots, \tilde{\xi}^j) = c^\pi \left(\iota(\tilde{\xi}^1) \cdots \iota(\tilde{\xi}^j) \omega \right),$$

where the ξ^k are vector fields on Y , ι denotes contraction, and $c^\pi(\cdot)$ denotes Clifford multiplication on $\mathbb{S}(X/Y)$ by the vertical part of the argument.

With this understood, we may define the pushforward for a general \mathbb{Z}_2 -graded vector bundle over a Riemannian family, with superconnection.

Definition 6.18. Let $\pi : X \rightarrow Y$ be a Riemannian family with compact and spin fibres. Let $E \rightarrow X$ be a finite dimensional, complex, hermitian, \mathbb{Z}_2 -graded vector bundle with superconnection ∇ . We define the pushforward \mathbb{Z}_2 -graded vector bundle $\pi_! E \rightarrow Y$ with superconnection $\pi_! \nabla$ as follows:

- $\pi_! E$: Define $\pi_! E \rightarrow Y$ to be the infinite dimensional, \mathbb{Z}_2 -graded, complex, hermitian vector bundle with typical fibre $(\pi_! E)_y = \Gamma(\mathbb{S}(X/Y)_y \otimes E_y)$.
- $\pi_! \nabla$: Write $\nabla = \nabla^E + \omega$, where ∇^E is an ordinary connection on E .

Then

$$\pi_! \nabla = \mathcal{D}^\pi(\nabla^E) + \nabla^{\pi_! E} + \frac{1}{4} c^\pi(T^\pi) + \pi_! \omega.$$

Here

$$\nabla^{\pi_! E} = \nabla^{\pi_! \mathbf{1}} \otimes 1 + 1 \otimes \nabla^E.$$

Remark 6.19. If ∇ is unitary, so is $\pi_! \nabla$.

Remark 6.20. We see that $[\pi_! \nabla]_{(0)} = \mathcal{D}^\pi(\nabla)$.

Remark 6.21. As required, if Y is spin, then with any choice of Riemannian structure on Y , $\mathcal{D}_Y(\pi_! \nabla) = \mathcal{D}_X(\nabla)$.

Remark 6.22. If $Y = \{\text{pt}\}$, we have $\pi_! \nabla = \mathcal{D}(\nabla)$, and if $X = \{\text{pt}\}$, $\pi_! E = E$ and $\pi_! \nabla = \nabla$.

Remark 6.23. The \mathbb{R}^\times -action on Riemannian maps (Eq. 6.8), and the \mathbb{R}^\times -action on superconnections (Eq. 2.12) intertwines:

$$\pi_!^t \nabla = (\pi_! \nabla^{1/t})^t. \tag{6.24}$$

Remark 6.25. The superconnection may be very suggestively written in terms of the degenerate Clifford module $\mathbb{S}_0 = \pi^*\Omega(Y) \otimes \mathbb{S}(X/Y)$ (discussed in Rem. 6.10), which we recall is a module for $\text{a-lim } \text{Cl}(X)$. In terms of the degenerate Clifford multiplication, the pushed forward superconnection may be written

$$\pi_!\nabla = c_0(\text{a-lim } \nabla^X \oplus \nabla),$$

where $\text{a-lim } \nabla^X$ is the adiabatic limit of the Levi-Civita connection on X . In this way, $\pi_!\nabla$ may be regarded as the adiabatic limit of the Dirac operator coupled to ∇ on X .

6.3 The Chern character

Let $\pi : X \rightarrow Y$ be a Riemannian family, with compact and spin fibres, and $E \rightarrow X$ be a finite dimensional, complex, hermitian, \mathbb{Z}_2 -graded vector bundle with unitary superconnection ∇ . The Chern character form associated with $\pi_!\nabla$ is defined by

$$\text{ch } \pi_!\nabla = \text{Tr } e^{-(\pi_!\nabla)^2}. \quad (6.26)$$

Some work is needed to understand Eq. 6.26, and to show that the Chern character is well defined. Details may be found in the appendix of Ch. 9 in [10]. We sketch the main points. A priori, we have $(\pi_!\nabla)^2 \in \Omega(Y; \text{End}(\pi_!E))^{\text{ev}}$. Calculation shows that at $x \in X$, $(\pi_!\nabla)^2$ is a local second order differential operator acting on $\mathbb{S}(X/Y) \otimes E$, with coefficients in $\bigwedge T^*(Y)_x$, of the form

$$(\pi_!\nabla)^2 = \mathcal{D}^\pi(\nabla)^2 + N_x,$$

where N_x is a nilpotent first order differential operator, its coefficients being of strictly positive degree. The elliptic theory developed in Ch. 3 shows $e^{-\mathcal{D}^\pi(\nabla)^2}$ to be trace class. Applying Duhamel's formula to $e^{-(\pi_! \nabla)^2}$ yields

$$e^{-(\pi_! \nabla)^2} = \sum_{i \geq 0} (-1)^i \left[e^{-\mathcal{D}^\pi(\nabla)^2} N_x \right]^{\#k} \# e^{-\mathcal{D}^\pi(\nabla)^2},$$

where

$$e^A B \# e^C = \int_0^1 ds e^{sA} B e^{(1-s)C}. \quad (6.27)$$

The nilpotency of N_x makes the sum finite, showing that $\text{ch}(\pi_! \nabla) \in \Omega(Y)$ is indeed well defined.

Some remarks further remarks on the nature of the Chern character and the curvature are in order. We have seen that $\text{Curv}(\pi_! \nabla)$ is a second order elliptic differential operator on each fibre. The operator

$$e^{-t(\pi_! \nabla)^2}$$

is the semigroup associated to heat operator

$$\frac{\partial}{\partial t} + (\pi_! \nabla)^2 \quad (6.28)$$

acting on elements of $\wedge T^* Y_y \otimes (\pi_! E)_y$, $y \in Y$, where we recall that elements of $(\pi_! E)_y$ are sections of $(\mathbb{S}(X/Y) \otimes E)|_{\pi^{-1}\{y\}} \rightarrow \pi^{-1}\{y\}$. It is a smoothing operator, so represented by a family of smooth integral kernels $p(t; x, x') : (\wedge T^* Y) \otimes E_x \rightarrow (\wedge T^*(Y)) \otimes E_{x'}$ (only defined when $\pi(x) = \pi(x') = y$). The Chern character, then, is computed by

$$(\text{ch } \pi_! \nabla)(y) = \int_{\pi^{-1}\{y\}} dx \text{Tr } p(1; x, x). \quad (6.29)$$

6.4 A families index theorem

We are now in a position to state and prove the main theorem of this section.

Theorem 6.30. *Let $\pi : X \rightarrow Y$ be a Riemannian family, with compact and spin fibres of dimension n . Let $E \rightarrow X$ be a finite dimensional, complex, hermitian, \mathbb{Z}_2 -graded vector bundle with unitary superconnection ∇ . Then*

$$\lim_{t \rightarrow 0} \text{ch } \pi_!^t \nabla = (2\pi i)^{-n/2} \pi_* \hat{A}(\Omega^\pi) \text{ch}(\nabla).$$

We will in fact prove a “local” version of the theorem. We recall the discussion in Sec. 6.3 – the curvature of the superconnection $\pi_! \nabla$ is a second order elliptic differential operator on the fibres, acting on sections of $\wedge T_y^* Y \otimes (\mathbb{S}(X/Y) \otimes E)|_{\pi^{-1}\{y\}}$ for a fibre over $y \in Y$. The Chern character is a smoothing operator on these sections, and thus represented by an integral kernel:

$$\exp[-t(\pi_! \nabla^s)^2] \phi(\cdot) = \int_{\pi^{-1}\{y\}} dx p_{t,s}(\cdot, x) \phi(x).$$

Proposition 6.31. *Let $p_{t,s}(x, y)$ be the integral kernel for $\exp[-t(\pi_! \nabla^s)^2]$.*

Then there exists a small t asymptotic expansion

$$p_{t,s/t}(x, x) \sim (4\pi t)^{-n/2} \sum_{i \geq 0} t^i A_i(x),$$

where

$$1. A_i(x) \in \sum_{j \leq 2i} \Omega^j(X; \text{End}_{\text{Cl}(X/Y)}(E)),$$

2. The full symbol of $p_{t,s/t}(x, x)$, defined by

$$\sigma(p) = \sum_{i=0}^{\dim X/2} [\sigma(A_i)]_{(2i)},$$

where we recall σ on the right is the symbol map defined in Eq. 6.12, is given by

$$\sigma(p) = \hat{A}(\Omega^\pi) \exp(-\nabla^2). \quad (6.32)$$

We note that Th. 4.1 is recovered by setting $Y = \{\text{pt}\}$. We begin by showing how prop. 6.31 implies Th. 6.30, essentially following the argument in [10] Ch.10.

Proof of Th. 6.30. Let $U_\varepsilon : \Omega(Y) \rightarrow \Omega(Y)$ be given by

$$U_\varepsilon(\alpha) = \varepsilon^{-k} \alpha, \quad (6.33)$$

for $\alpha \in \Omega^k(Y)$. We compute that the integral kernel for $\exp[-(\pi_!^t \nabla)^2]$ is given by

$$U_{t^{1/2}} p_{t,s/t}(x, y), \quad (6.34)$$

where we recall that the \mathbb{R}^\times -action on the Riemannian map and the superconnection are related by Eq. 6.24. Thus

$$\text{ch } \pi_!^t \nabla = \int_{X/Y} dx \text{ Tr } U_{t^{1/2}} p_{t,s/t}(x, x),$$

where the supertrace on the right is on $\text{End}(S \otimes E)$. For small t , then,

$$\text{ch } \pi_!^t \nabla \sim (4\pi t)^{-n/2} \int_{X/Y} dx \text{ Tr } \sum_i t^i U_{t^{1/2}} A_i(x, x).$$

One may compute the supertrace of $\alpha \in \Omega(Y) \otimes \text{End}(\mathbb{S}(X/Y) \otimes E)$ in terms of the bigrading on $\Omega(X)$ (Eq. 6.13) and the symbol map (Eq. 6.12) as

$$\text{Tr}(\alpha) = (2i)^{n/2} \sum_k \text{Tr}[\sigma(\alpha)]_{(k,n)},$$

where the supertrace on the right is on $\text{End}(E)$. Thus

$$\text{ch } \pi_!^t \nabla \sim (2\pi i)^{-n/2} \int_{X/Y} dx \sum_{j,k} t^{j-(n+k)/2} \text{Tr}[\sigma(A_j(x, x))]_{(k,n)}.$$

This converges as $t \rightarrow 0$, as $[\sigma(A_j)]_{(k,n)} = 0$ if $2j < k + n$, and the explicit formula for the full symbol (Eq. 6.32) gives the result. \square

The proof of Prop. 6.31 is very similar to that of Th. 4.1, and we will proceed quickly, exposing the differences, but not dwelling unnecessarily on the details. The general strategy is to follow the proof of Prop. 6.31, but instead of working on the entire manifold, to work on a point on the fibre, and examine the heat kernel associated to $\pi_! \nabla$ in a neighbourhood of that point on the fibre.

We begin with an analogue of the Weitzenböck formula.

Lemma 6.35. *The curvature of the superconnection $\pi_! \nabla$ is given by*

$$\text{Curv}(\pi_! \nabla) = (\nabla^\pi)^* \nabla^\pi + \frac{1}{4} R^\pi + Q_0,$$

where $(\nabla^\pi)^* \nabla^\pi$ is the covariant laplacian built from the fibre Levi-Civita connection, R^π is the fibre scalar curvature, and

$$Q_0 = c_0(\Omega^E) + [c_0(\nabla), c_0(\omega)] + c_0(\omega)^2,$$

where we have written $\nabla = \nabla + \omega$.

Proof. We recall that the Bismut superconnection may be thought of the Dirac operator on X (Rem. 6.25), and take the adiabatic limit of the usual Weitzenböck formula (3.1). This shows the result. \square

Let us now choose $x \in X$, with $y = \pi(x)$. Denote the fibre over y by X_y , and choose r such that the exponential map

$$\exp : B_r(0) \subset T_x(X_y) \rightarrow U \subset X_y$$

is a diffeomorphism of the ball of radius r to an open neighbourhood of x . As in the proof 4.1, we pull back geometry on U to $T_x(X_y) \cong (T_x(X/Y))$, and extend it to the whole of $T_x(X_y)$. We use the same symbols to denote the geometry on U and $T_x(X_y)$. We choose an orthonormal framing $\{e^i\}$ of $T_x(X_y)$ and a framing $\{f_\alpha\}$ of $T^*(Y)$. Using parallel transport along radial geodesics, we trivialise the geometry on $T_x(X_y)$. We may thus identify the degenerate Clifford algebra $\text{Cl}_0(X)$ with $\bigwedge^\bullet(T_x^*X)$. Care must be taken with the multiplication as $\text{a-lim } \nabla^X$ is not diagonal, and mixes the e_i and f_α . The following lemma (Lemma 10.25 in [10]) shows that this mixing is mild.

Lemma 6.36. *The action of the degenerate Clifford algebra $\bigwedge^\bullet(T_x^*X)$ on $\mathbb{S}_0(X) \cong \bigwedge^\bullet T_y^*Y \otimes \mathbb{S}(X/Y)$ (cf. Rem. 6.10), denoted by $c_0(\cdot)$ is given by*

$$\begin{aligned} c_0(e_i) &= c(e_i) + \sum_{\alpha} u_i^{\alpha} e(f_{\alpha}), \\ c_0(f_{\alpha}) &= e(f_{\alpha}), \end{aligned}$$

on U , where $c(\cdot)$ is Clifford multiplication on $\mathbb{S}(X/Y)$, $e(\cdot)$ is exterior multiplication, and the $u_i^{\alpha}(x') \in \mathbb{C}^{\infty}(U)$ are $O(|x'|)$.

The proof is given in [10], and is a simple computation in the radial coordinates – the functions u^α come from the fact that $(\text{a-lim } \nabla^X)e^i \neq 0$.

We now introduce an algebraic deformation $\hat{U}_\varepsilon : \bigwedge T_x^*(X) \rightarrow \bigwedge T_x^*(X)$ analogous to that in 2.4. It is defined by

$$\hat{U}_\varepsilon : \hat{a} \mapsto \varepsilon^{-|\hat{a}|} \hat{a},$$

on homogeneous elements, and, identifying $\text{Cl}(X/Y)$ with $\bigwedge T^*(X/Y)$, we extend this to a map $U_\varepsilon : T_y^*(Y) \otimes \text{Cl}(X_y) \rightarrow \bigwedge T_x^* X$. We also introduce the geometric deformation $T_\varepsilon : T(X_x) \rightarrow T(X_x)$ as follows

$$T_\varepsilon : \xi \mapsto \varepsilon \xi.$$

Defining $S_\varepsilon = U_\varepsilon T_\varepsilon^*$, we see easily that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{|\hat{a}|} S_\varepsilon c_0(a) S_\varepsilon^{-1} = e(\hat{a}) \quad (6.37)$$

for homogeneous $a \in \text{Cl}(X_y) \rightarrow \bigwedge T_x^*$. Define the family of heat operators

$$\begin{aligned} H_\varepsilon &= \frac{\partial}{\partial t} + P_\varepsilon, \\ P_\varepsilon &= S_\varepsilon \left(\pi_! \nabla^{s/\varepsilon^2} \right)^2 S_\varepsilon^{-1}. \end{aligned}$$

The following lemma is key.

Lemma 6.38. *The family of operators P_ε has a limit as $\varepsilon \rightarrow 0$. In the framing above, it is given by*

$$P_0 = \lim_{\varepsilon \rightarrow 0} P_\varepsilon = - \sum_{i,j} \left(\frac{\partial}{\partial \xi^i} - \frac{1}{4} e \left(\Omega_{ij}^{X/Y}(x) \right) \xi^j \right)^2 + e \left((\nabla^s)^2 \right) (x),$$

where the ξ^i are coordinates on $T_x(X_y)$.

Proof. This lemma is proved entirely analogously to 3.34, where 6.35 plays the role of 3.1, the limit

$$\lim_{\varepsilon \rightarrow 0} S_\varepsilon (\nabla^\pi)^* \nabla^\pi S_\varepsilon^{-1}$$

is computed in [10], Ch. 10.4, and

$$\lim_{\varepsilon \rightarrow 0} S_\varepsilon Q_0 S_\varepsilon^{-1} = e \left((\nabla^s)^2 \right) (x)$$

from 6.37. □

Following the proof of 4.1, we next compute the heat kernel for H_0 .

Proposition 6.39. *The heat kernel for the operator*

$$H_0 = \frac{\partial}{\partial t} - \sum_{i,j} \left(\frac{\partial}{\partial \xi^i} - \frac{1}{4} e(\Omega_{ij}^{X/Y}(x) \xi^j) \right)^2 + e \left((\nabla^s)^2 \right) (x)$$

at the origin is given by the formula

$$\tilde{p}_{t,s}^0(0,0) = (4\pi t)^{-n/2} \sqrt{\det \left(\frac{t\Omega^\pi/2}{\sinh(t\Omega^\pi/2)} \right)} e^{-t(\nabla^s)^2}.$$

We note that the small t asymptotic expansion of $p_{t,s/t}^0(0,0)$ is precisely as in 6.31.

Let us now define

$$\tilde{p}_{s,t}^\varepsilon(x,y) = \varepsilon^n S^\varepsilon p_{\varepsilon^2 t, s/\varepsilon^2}(x,y)$$

and note that this is annihilated by H_ε and converges to $\delta_x(y)$ as $t \rightarrow 0$ in H_k for all $k < -n/2$. By uniqueness of solutions this is then the heat kernel for H_ε . We now argue precisely as in the proof of 4.1. We write

$$p_{t,s}(x,y) = \sum_{I,J} p_{t,s}^{IJ}(x,y) e^I f^J,$$

where I, J are multi-indices. Then

$$\tilde{p}_{t,t}^\varepsilon(x, s) = \sum_{I,J} \varepsilon^{n-|I|-|J|} p_{\varepsilon^2 t, s/\varepsilon^2}^{IJ}(0, 0) \hat{e}^I f^J.$$

An easy extension to the argument in Sec. 3.3.2 show that $p_{t,s/t}(0, 0)$ has a small t asymptotic expansion of the form

$$p_{t,s/t}(x, x) \sim t^{-n/2} \sum_{j,I,J} t^j A_j^{IJ} e^I f^J.$$

Thus

$$\tilde{p}_{t,s/t}^\varepsilon(x, x) \sim \sum_{j,I,J} t^{j-n/2} \varepsilon^{2j-|I|-|J|} A_j^{IJ}.$$

We see that $A_j^{IJ} = 0$ if $2j < |I| + |J|$, and that

$$\sigma(p) = \sum_{j=0}^{\dim X/2} \sum_{\substack{I,J \\ |I|+|J|=2j}} A_j^{IJ} = \hat{A}(\Omega^\pi) \exp(-(\nabla^s)^2),$$

as the small t asymptotic expansion of $\tilde{p}_{t,s/t}^\varepsilon$ converges to that of $\tilde{p}_{t,s/t}^0$ as $\varepsilon \rightarrow 0$.

This then proves the proposition.

6.4.1 A transgression formula for the Bismut superconnection

We now prove an important corollary of Th. 6.30. Let $\pi : X \rightarrow Y$, $E \rightarrow X$ and ∇ be as before. The transgression formula for superconnections shows [10], for $0 < t < T$,

$$\mathrm{ch} \pi_!^T \nabla - \mathrm{ch} \pi_!^t \nabla = -\mathrm{d} \int_t^T \mathrm{d}s \, \mathrm{Tr} \left(\frac{\partial \pi_!^s \nabla}{\partial s} e^{-(\pi_!^s \nabla)^2} \right).$$

An immediate consequence of Th. 6.30 is that the left hand side converges as $t \rightarrow 0$. We wish to show more.

Theorem 6.40. 1. *The differential form*

$$\alpha(t) = \text{Tr} \left(\frac{\partial \pi_{\dagger}^t \nabla}{\partial s} e^{-(\pi_{\dagger}^t \nabla)^2} \right)$$

has an asymptotic expansion as $t \rightarrow 0$ of the form

$$\alpha(t) \sim \sum_{j \geq 1} t^{j/2-1} \alpha_{j/2},$$

with $\alpha_{j/2} \in \Omega(B)$.

2.

$$\text{ch } \pi_{\dagger}^t \nabla = (2\pi i)^{-n/2} \pi_* \hat{A}(\Omega^\pi) \text{ch } \nabla - d \int_0^t ds \alpha(s),$$

where $n = \dim X/Y$.

Proof. The proof follows [10] Ch. 10.5 very closely. Part 2 in the statement is an immediate consequence of part 1. We begin by defining the family $\tilde{\pi} : \tilde{X} \rightarrow \tilde{Y}$, where $\tilde{X} = X \times (0, \infty)$, and \tilde{Y} is defined similarly. We make $\tilde{\pi}$ a Riemannian family by extending the projection P trivially, but endowing the fibres with the vertical metric

$$g_{\tilde{X}/\tilde{Y}} = s^{-1} g_{X/Y},$$

where $s \in (0, \infty)$. We extend E trivially to form $\tilde{E} \rightarrow \tilde{X}$ and give it the superconnection

$$\tilde{\nabla} = ds \partial_s + \nabla.$$

A short computation shows

$$\tilde{\pi}_{\dagger} \tilde{\nabla} = \pi_{\dagger}^s \nabla + ds \partial_s - \frac{n}{4s} ds,$$

where $n = \dim X/Y$, so that

$$(\tilde{\pi}_! \tilde{\nabla})^2 = (\pi_!^s \nabla)^2 + \frac{\partial \pi_!^s \nabla}{\partial s} ds.$$

Thus

$$\text{ch } \tilde{\pi}_! \tilde{\nabla} = \text{ch } \pi_!^s \nabla - \alpha(s) ds.$$

By Prop. 6.31, we have the small t asymptotic expansion

$$\text{ch } \tilde{\pi}_!^t \tilde{\nabla} \sim \sum_{j \geq 0} t^{j/2} \tilde{a}_{j/2},$$

with $\tilde{a}_{j/2} \in \Omega^\bullet(\tilde{Y})$. On the other hand,

$$\text{ch } \tilde{\pi}_!^t \tilde{\nabla} = \text{ch } \pi_!^{st} \nabla - \text{Tr} \left(\frac{\partial \pi_!^{st} \nabla}{\partial s} e^{-(\pi_!^{st} \nabla)^2} \right) ds,$$

which allows us to write

$$\tilde{a}_{j/2} = a_{j/2} - \alpha_{j/2} ds.$$

Here the $a_{j/2}, \alpha_{j/2} \in \rho^* \Omega(Y)$, where $\rho : \tilde{Y} \rightarrow Y$ is the obvious projection. The chain rule shows

$$\left. \frac{\partial \pi_!^{st} \nabla}{\partial s} \right|_{s=1} = t \frac{\partial \pi_!^t \nabla}{\partial t},$$

which shows

$$\alpha(t) \sim \sum_{j \geq 0} t^{j/2-1} \alpha_{j/2}(x, s=1)$$

for small t . We need to show α_0 vanishes. Th. 6.30 asserts

$$a_0 - \alpha_0 ds = (2\pi i)^{-n/2} \tilde{\pi}_* \hat{A}(\Omega^\pi) \text{ch } \tilde{\nabla}.$$

On the other hand,

$$\tilde{\pi}_* \hat{A}(\Omega^\pi) \text{ch } \tilde{\nabla} = \rho^* \pi_* \hat{A}(\Omega^\pi) \text{ch } \nabla,$$

so cannot involve ds . The result is thus demonstrated. \square

Chapter 7

Determinant line bundles

In this chapter we construct a graded line bundle associated with a family of Dirac operators coupled to superconnections. The basic construction will follow that of Quillen [30] and Bismut and Freed [13, 14], and reduces to theirs in the case that the superconnection is an ordinary connection. We will see, however, that the connection we obtain for a Dirac operator coupled to a general superconnection is different from that produced in their construction of a geometric determinant line bundle for a general family of Dirac operators.

Given a Riemannian map $\pi : X \rightarrow Y$, with compact and spin fibres, and a finite dimensional, complex and hermitian \mathbb{Z}_2 -graded vector bundle $E \rightarrow X$, with unitary superconnection ∇ , we shall, in a functorial way, construct a \mathbb{Z}_2 -graded hermitian line bundle $\det(\pi_! \nabla) \rightarrow Y$ with section and connection, such that the section vanishes precisely at points $y \in Y$ where $\mathcal{D}^\pi(\nabla)$ is not invertible, and such that the curvature is given by the formula

$$\text{Curv}(\det(\pi_! \nabla)) = (2\pi i)^{-n/2} \left[\pi_* \hat{A}(\Omega^\pi) \text{ch}(\nabla) \right]_{(2)},$$

where n is the dimension of the fibre. We will see that the holonomy of the determinant line bundle is computed by the τ -invariant discussed in Ch. 5.

7.1 Determinant line bundles: the finite dimensional case

Before embarking on the construction of the determinant line bundle in general, we will examine a finite dimensional model. Let Y be a compact manifold, and $E \rightarrow Y$ be a complex, hermitian, and finite dimensional \mathbb{Z}_2 -graded vector bundle, with unitary superconnection ∇ . We wish to construct the “determinant line bundle” $\det(\nabla) \rightarrow Y$, which will be a \mathbb{Z}_2 -graded hermitian line bundle, with section and connection. The section will vanish precisely where $[\nabla]_{(0)}$ is not invertible, and the connection will have curvature

$$\text{Curv}(\det(\nabla)) = [\text{ch } \nabla]_{(2)}.$$

We define $\det(\nabla) \rightarrow Y$ as the line bundle

$$\det(\nabla) = \det(E^1) \otimes \det(E^0)^{-1}, \quad (7.1)$$

where the determinant line bundles on the right hand side are the usual determinants of vector bundles: $\det(E^i) = \bigwedge^{\dim E^i} (E^i)$. This is a \mathbb{Z}_2 -graded line bundle, with degree $\dim E^1 - \dim E^0 \pmod 2$ on each component of Y . It is also hermitian, inheriting a metric from those on the E^i .

The determinant line bundle has a natural section. It will be convenient to fix some notation. We decompose the superconnection as:

$$\nabla = \mathcal{D} + \nabla + \sum_{i \geq 2} \omega_i, \quad (7.2)$$

where $\mathcal{D} \in \Omega^0(Y; \text{End}(E))$, and $\omega_i \in \Omega^i(Y; \text{End}(E))$. By the unitarity of ∇ , we may write

$$\mathcal{D} = \begin{pmatrix} & \not{D}^* \\ \not{D} & \end{pmatrix}$$

where $\not{D} : E^0 \rightarrow E^1$.¹ We note that $\text{Hom}(E^0, E^1) \cong E^1 \otimes (E^0)^*$, so that $\not{D} \in \Gamma(E^1 \otimes (E^0)^*)$. It thus induces a section $\det \not{D}$ of $\det(\nabla)$ that vanishes precisely when \mathcal{D} is not invertible. This is the section we seek. The grading of $\det(\nabla)$ may also be seen in terms of \not{D} : on any component it is given by $\text{index } \not{D} \pmod{2}$.

It remains to construct a unitary connection on $\det(\nabla)$ with appropriate curvature. Before doing so, we make a digression to derive a useful formula. Let us recall the transgression formula 2.14 applied to a family of superconnections $\nabla_t = \nabla + L_t$, $L_t \in \Omega(Y; \text{End } E)^{\text{odd}}$:

$$\frac{d}{dt} \text{ch } \nabla_t = -d \text{Tr} \left(e^{-\nabla_t^2} \dot{L}_t \right), \quad (7.3)$$

where “ \cdot ” means $\frac{d}{dt}$. Restricting our attention to the two-form part of the Chern character, and applying the Duhamel formula, we obtain the equation

$$\begin{aligned} \frac{d}{dt} [\text{ch } \nabla_t]_{(2)} = d \Bigg\{ & - \text{Tr} e^{-[L_t]_{(0)}^2} \left[\dot{L}_t \right]_{(1)} - \text{Tr} \left[e^{-L_t^2} \right]_{(1)} \left[\dot{L}_t \right]_{(0)} \\ & + \text{Tr} e^{-[L_t]_{(0)}^2} \nabla [L_t]_{(0)} \# e^{-[L_t]_{(0)}^2} \left[\dot{L}_t \right]_{(0)} \Bigg\}, \quad (7.4) \end{aligned}$$

¹The notation is chosen to make the analogy with the determinant line bundle associated to families transparent: we are to think of the degree zero part of ∇ as the fibrewise Dirac operator for the trivial family.

where we recall (Eq. 6.27) that

$$e^A B \# e^C \equiv \int_0^1 ds e^{sA} B e^{(1-s)C}.$$

An immediate consequence of Eq. 7.4 is that the two-form component of the Chern character of a superconnection is independent of all terms of the superconnection of cohomological degree greater than or equal to two.

We wish now to write $[\text{ch } \nabla]_{(2)}$ in terms of $\text{Curv}(\nabla)$. To do this, we apply Eq. 7.4 to the family ∇^t (obtained from the usual \mathbb{R}^\times -action on superconnections in Eq. 2.12):

$$\begin{aligned} [\text{ch } \nabla]_{(2)} - [\text{ch } \nabla]_{(2)} &= \int_0^1 dt \frac{d}{dt} [\text{ch } \nabla^t]_{(2)} \\ &= \frac{1}{2} d \int_0^1 dt \text{Tr} \left[e^{-t\mathcal{D}^2} \nabla \mathcal{D} \# e^{-t\mathcal{D}^2} \mathcal{D} \right] \\ &= \frac{1}{2} d \int_0^1 dt \text{Tr} \left[\nabla \mathcal{D} e^{-t\mathcal{D}^2} \mathcal{D} \right] \\ &= \frac{1}{2} d \text{Tr} \left[\left(1 - e^{-\mathcal{D}^2} \right) \nabla \mathcal{D} \mathcal{D}^{-1} \right]. \end{aligned} \quad (7.5)$$

There are several things that may appear problematic in the calculation. The first concern is the possibility of divergences in the first line of the calculation. These can only enter in the terms in the superconnection of cohomological degree greater than or equal to two, so do not contribute. The second apparent concern is the appearance of \mathcal{D}^{-1} : it appears that \mathcal{D} needs to be invertible to make sense of the expression. However, expanding the exponential in Eq. 7.5 as a power series in \mathcal{D} , we see that the leading order term in Eq. 7.5 is in fact degree one in \mathcal{D} , and in this sense the last line is perfectly well defined even when \mathcal{D} is not invertible.

The curvature of the connection induced on $\det(\nabla)$ by ∇ is precisely $[\text{ch } \nabla]_{(2)}$. Denoting the connection induced by ∇ on $\det(\nabla)$ by $\nabla^{\det(\nabla)}$, we define the connection

$$\nabla^{\det(\nabla)} = \nabla^{\det(\nabla)} + \frac{1}{2} \text{Tr} \left[\left(1 - e^{-\mathcal{D}^2} \right) \nabla \mathcal{D} \mathcal{D}^{-1} \right]. \quad (7.6)$$

This is a unitary connection, as $\nabla^{\det(\nabla)}$ is, and

$$\frac{1}{2} \text{Tr} \left[\left(1 - e^{-\mathcal{D}^2} \right) \nabla \mathcal{D} \mathcal{D}^{-1} \right]$$

is imaginary. It also has the required curvature:

$$\text{Curv}(\nabla^{\det(\nabla)}) = \text{Curv}(\nabla^{\det(\nabla)}) + \frac{1}{2} d \text{Tr} \left[\left(1 - e^{-\mathcal{D}^2} \right) \nabla \mathcal{D} \mathcal{D}^{-1} \right] = [\text{ch } \nabla]_{(2)}.$$

We have thus constructed a geometric line bundle $(\det(\nabla), \nabla^{\det(\nabla)}) \rightarrow X$ with canonical section $\det \not{D}$. Furthermore, this construction is natural, in the sense that it commutes with pullbacks. The curvature of $\det(\nabla)$ is computed by

$$\text{Curv}(\nabla^{\det(\nabla)}) = [\text{ch } \nabla]_{(2)}.$$

7.2 Determinant line bundles for a family of Dirac operators

We now wish to examine the following situation: let $\pi : X \rightarrow Y$ be a Riemannian map with compact and spin fibres of dimension n , and $E \rightarrow X$ be a complex, hermitian and finite dimensional \mathbb{Z}_2 -graded vector bundle with unitary superconnection ∇ . We wish to associate to this data a complex and hermitian \mathbb{Z}_2 -graded line bundle $\det(\pi_! \nabla) \rightarrow Y$, with a section that vanishes

precisely where the fibre Dirac operator $\mathcal{D}^\pi(\nabla)$ is not invertible, and connection with curvature $[(2\pi i)^{-n/2} \hat{A}(\Omega^\pi) \text{ch } \nabla]_{(2)}$. The construction summarised in Th. 7.1 may be seen as a special case of this where $X = Y$ and $\pi : X \rightarrow Y$ is the identity. Our construction will generalise that of Quillen [30] and Bismut and Freed [13, 14]. It should be noted that the construction of Bismut and Freed in [13, 14] holds for general families of first order elliptic differential operators, and in particular, for Dirac operators coupled to superconnections. However, the connection that their construction yields for a Dirac operator coupled to a general superconnection does not have a curvature computed by natural geometric quantities. The essential novelty of our construction is the construction of a natural connection on the determinant line bundle, with curvature computed by $[(2\pi i)^{-n/2} \hat{A}(\Omega^\pi) \text{ch } \nabla]_{(2)}$. It will turn out that the holonomy of the determinant line bundle is computed by the adiabatic limit of an η -invariant.

7.2.1 The bundle with its section

We begin by construction the line bundle, with its section. It will be convenient to introduce some notation. We write

$$\pi_! \nabla = \mathcal{D}^\pi + \nabla + \sum_{i \geq 2} \omega_i,$$

with $\mathcal{D}^\pi \in \Omega^0(Y; \text{End}(\pi_! E))$ is the fibre Dirac operator coupled to ∇ ,² ∇ is a connection on $\pi_! E$, and $\omega_i \in \Omega^i(Y; \text{End}(\pi_! E))$. We write

$$\mathcal{D}^\pi = \begin{pmatrix} \mathcal{D}^\pi & \mathcal{D}^{\pi*} \end{pmatrix}$$

so that $\mathcal{D}_y^\pi \in \text{Hom}(\pi_! E^0, \pi_! E^1)_y$ at any $y \in Y$. We recall from Ch. 3 that the spectrum of $\mathcal{D}_y^{\pi*} \mathcal{D}_y^\pi$ at any $y \in Y$ is non-negative, and discrete. We define an open cover $\{U_a\}_{a \in \mathbb{R}}$ of Y by

$$U_a = \{y \in Y : a \notin \text{spec } \mathcal{D}_y^{\pi*} \mathcal{D}_y^\pi\},$$

and define the projections $\Pi_{(a,b)} \in \Gamma(\text{Hom}(\pi_! E))$ which, at each $y \in Y$, projects $(\pi_! E)_y$ to the subspace spanned by eigenvectors with eigenvalues in (a, b) . We will denote the image of $\Pi_{(a,b)}$ by $(\pi_! E)_{(a,b)}$, and allow $a = -\infty, b = \infty$. Notice that $\Pi_{(a,b)}$ respects the grading of $\pi_! E$, and if b is bounded, $(\pi_! E)_{(a,b)}$ is finite dimensional at each $y \in Y$.³

Let us now restrict our attention to U_a . Over U_a the bundle $\pi_! E$ decomposes as a direct sum

$$\pi_! E = (\pi_! E)_{(-\infty, a)} \oplus (\pi_! E)_{(a, \infty)}, \quad (7.7)$$

and that \mathcal{D}^π respects the decomposition, restricting to a linear map

$$\mathcal{D}_{(-\infty, a)}^\pi : (\pi_! E)_{(-\infty, a)}^{\text{ev}} \rightarrow (\pi_! E)_{(-\infty, a)}^{\text{odd}}. \quad (7.8)$$

²Throughout this section we suppress the explicit dependence of the Dirac operator on the superconnection.

³Of course the laplacians have a non-negative spectrum, so one may replace $\Pi_{(a,b)}$, $a < 0$, with $\Pi_{[0,b)}$ without loss of generality.

We may thus define the \mathbb{Z}_2 -graded line bundle $\det(\pi_! \nabla) \rightarrow U_a$ as

$$\det(\pi_! \nabla)_{(-\infty, a)} = \left(\det(\pi_! E)_{(-\infty, a)}^{\text{odd}} \right) \otimes \left(\det(\pi_! E)_{(-\infty, a)}^{\text{ev}} \right)^*,$$

where we remember that “det” on the right hand side means the top exterior power. The degree of the line bundle on any component of U_a is given by the mod 2 difference of dimensions of the even and odd components of $\pi_! E$ at any point in the component. As in the finite dimensional case,

$$\mathcal{D}_{(-\infty, a)}^\pi \in \Gamma \left((\pi_! E)_{(-\infty, a)}^{\text{odd}} \otimes ((\pi_! E)_{(-\infty, a)}^{\text{ev}})^* \right)$$

induces a section $\det \mathcal{D}_{(-\infty, a)}^\pi \in \Gamma \left(\det(\pi_! \nabla)_{(-\infty, a)} \right)$.

We wish to extend the line bundle and section to the whole of Y . Let us concentrate on the overlap $U_a \cap U_b$, and assume without loss of generality that $a < b$. On this open set $\pi_! E$ decomposes as

$$\pi_! E = (\pi_! E)_{(-\infty, a)} \oplus (\pi_! E)_{(a, b)} \oplus (\pi_! E)_{(b, \infty)},$$

with

$$(\pi_! E)_{(-\infty, b)} = (\pi_! E)_{(-\infty, a)} \oplus (\pi_! E)_{(a, b)}. \quad (7.9)$$

The fibrewise Dirac operator \mathcal{D}^π respects the decomposition, and is an isomorphism on $(\pi_! E)_{(a, b)}$. The line bundle $\det(\pi_! \nabla)_{(a, b)}$ is purely even, and has a canonical non-vanishing section $\det \mathcal{D}_{(a, b)}^\pi$, where everything is defined analogously to the line bundles in the previous paragraph. In the light of the decomposition 7.9, this gives an isomorphism of \mathbb{Z}_2 -graded line bundles

$$\det \mathcal{D}_{(a, b)}^\pi : \det(\pi_! \nabla)_{(-\infty, a)} \xrightarrow{\sim} \det(\pi_! \nabla)_{(-\infty, b)}$$

defined by

$$\sigma \mapsto \sigma \otimes \det \mathcal{D}_{(a,b)}^\pi.$$

These are the required transition functions, allowing us to define a \mathbb{Z}_2 -graded complex line bundle $\det(\pi_! \nabla) \rightarrow Y$, with section $\det \mathcal{D}^\pi$. On any component of Y , the degree of the line bundle is given by index $\mathcal{D}^\pi \bmod 2$.

Definition 7.10. The complex \mathbb{Z}_2 -graded line bundle $\det(\pi_! \nabla)$ with section $\det \mathcal{D}^\pi$ is the *determinant line bundle* associated to the data $(E, \nabla) \rightarrow X \rightarrow Y$.

7.2.2 The metric

We now wish to give $\det(\pi_! \nabla)$ a hermitian structure. Again, we concentrate on the open set U_a . The line bundle $\det(\pi_! \nabla)_{(-\infty, a)}$ has a natural hermitian structure, induced by the L^2 -metric on $\pi_! E$. Let us now examine how the metric of $\det(\pi_! (\nabla))_{(-\infty, a)}$ compares to $\det(\pi_! (\nabla))_{(-\infty, b)}$ on $U_a \cap U_b$ (again, and always, we assume $a < b$). Let $\sigma \in \det(\pi_! \nabla)$. By definition

$$\sigma|_b \cong \sigma|_a \otimes \det \mathcal{D}_{(a,b)}^\pi.$$

Thus

$$\|\sigma\|_{(-\infty, b)}^2 = \|\sigma\|_{(-\infty, a)}^2 \prod_{\substack{\lambda \in \text{spec } \mathcal{D}^* \mathcal{D} \\ a < \lambda < b}} \lambda.$$

We could remove this discrepancy if we could define a new metric on U_a by

$$\|\sigma\|^2 = \|\sigma\|_{(-\infty, a)}^2 \prod_{\substack{\lambda \in \text{spec } \mathcal{D}^* \mathcal{D} \\ a < \lambda}} \lambda, \quad (7.11)$$

and similarly on every other open set U_b . As written this makes no sense, but we may use ζ -function renormalisation to make sense of the formula. We will provide a sketch here, and point the reader to [13] for the detailed analysis.

Associated to a second order elliptic operator E one may define a ζ -function

$$\zeta[E](s) = \sum_{\lambda \in \text{spec } E} \lambda^s.$$

This is well defined for $\text{Re}(s)$ large enough, and admits an analytic continuation to a meromorphic function on \mathbb{C} . Differentiating the ζ -function in s , one sees

$$\zeta[E]'(s) = \sum_{\lambda \in \text{spec } E} \log(\lambda) \lambda^{s-1}.$$

It is thus reasonable to define

$$\prod_{\lambda \in \text{spec } E} \lambda \equiv e^{\zeta[E]'(1)},$$

as long as $\zeta[E](s)$ is analytic at $s = 1$. Bismut and Freed [13] show this to be the case for a general first order elliptic differential operator, and it is in this sense that we should interpret Eq. 7.11. With this interpretation, we may now use Eq. 7.11 to give $\det(\pi_! \nabla)$ a hermitian structure.

7.2.3 The connection

We wish to endow $\det(\pi_! \nabla) \rightarrow Y$ with a unitary connection $\nabla^{\det(\pi_! \nabla)}$, with curvature

$$\text{Curv}(\nabla^{\det(\pi_! \nabla)}) = (2\pi i)^{-n/2} \left[\pi_* \hat{A}(\Omega^\pi) \text{ch } \nabla \right]_{(2)}$$

where, as usual, $n = \dim X/Y$. We will do this in two stages: first, we will follow [13] to construct a connection that may be seen as that induced by $\nabla^{\pi_! E}$. We will then use similar arguments to those section 7.1 to modify this connection and produce the connection we require.

To begin we restrict our attention to U_a . The connection $\nabla^{\pi_! E}$ restricts to a connection $\nabla_{(-\infty, a)}^{\pi_! E}$ on $\det(\pi_! \nabla)_{(-\infty, a)}$. It is unitary with respect to the induced metric, but not with respect to the metric on $\det(\pi_! \nabla)$. As before, we examine the overlap $U_a \cap U_b$. Let $\sigma \in \Gamma(\det(\pi_! \nabla))$. We calculate:

$$\begin{aligned} \nabla_{(-\infty, b)}^{\pi_! E} \sigma|_b &= \nabla_{(-\infty, b)}^{\pi_! E} (\sigma|_a \otimes \det \not{D}_{(a, b)}^\pi) \\ &= \left(\nabla_{(-\infty, a)}^{\pi_! E} \sigma|_a \right) \otimes \det \not{D}_{(a, b)}^\pi \\ &\quad + \text{Tr}_0 \left[(\nabla \not{D}) \not{D}^{-1} \right]_{(a, b)} \sigma|_a \otimes \det \not{D}_{(a, b)}^\pi, \end{aligned}$$

where Tr_0 is the *ungraded* trace.

If we could interpret

$$\alpha_a = \text{Tr}_0 \left[(\nabla \not{D}) \not{D}^{-1} \right]_{(a, \infty)},$$

appropriately, we could define a connection on $\det(\pi_! \nabla)$ by

$$\nabla_{(-\infty, a)}^0 = \nabla_{(-\infty, a)}^{\pi_! E} + \alpha_a \tag{7.12}$$

on U_a . Bismut and Freed [13, 14] show how one may interpret α using ζ -function normalisation, and that the resultant connection is well defined, unitary, and has curvature

$$\text{Curv}(\nabla^0) = \text{LIM}_{t \rightarrow 0} \left[\text{ch}(\pi_! \nabla)^t \right]_{(2)}.$$

Here “LIM” is defined as follows. Suppose $f(t) \in C^\infty((0, \infty))$, with a small t asymptotic expansion

$$f(t) \sim \sum_{k \geq k_0} a_k t^k.$$

Then we define

$$\text{LIM}_{t \rightarrow 0} f(t) \equiv a_0. \quad (7.13)$$

We will now modify the connection in Eq. 7.12 to obtain the connection we desire. We will follow an argument similar to that in Sec. 7.1. We begin by introducing two ancillary one-forms:

$$\begin{aligned} \beta_1(t) &= \int_t^1 dt \left[\text{Tr} \left(e^{(\pi! \nabla)^t} \frac{d(\pi! \nabla)^t}{dt} \right) \right]_{(1)}, \\ \beta_2(t) &= \int_t^1 dt \left[\text{Tr} \left(e^{\pi! \nabla} \frac{d\pi! \nabla}{dt} \right) \right]_{(1)}. \end{aligned}$$

A calculation similar to that in Eq. 7.5 gives a closed expression for $\beta_1(t)$:

$$\beta_1(t) = \frac{1}{2} \text{Tr} \left[e^{-\mathcal{D}(\nabla^t)^2} \nabla \mathcal{D}(\nabla) \mathcal{D}^{-1}(\nabla) \right]_{(2)} - \frac{1}{2} \text{Tr} \left[e^{-\mathcal{D}(\nabla)^2} \nabla \mathcal{D}(\nabla) \mathcal{D}^{-1}(\nabla) \right]_{(2)}.$$

Unfortunately $\beta_2(t)$ may not be written so explicitly. The transgression formula shows immediately that

$$\begin{aligned} d\beta_1(t) &= [\text{ch } \pi! \nabla]_{(2)} - [\text{ch}(\pi! \nabla)^t]_{(2)}, \\ d\beta_2(t) &= [\text{ch } \pi! \nabla]_{(2)} - [\text{ch}(\pi! \nabla^t)]_{(2)}. \end{aligned}$$

Both of these one-forms are purely imaginary. We define the connection

$$\nabla^{\det(\pi! \nabla)} \equiv \nabla^0 + \text{LIM}_{t \rightarrow 0} \beta_1(t) - \lim_{t \rightarrow 0} \beta_2(t), \quad (7.14)$$

where we recall LIM is the formal limit defined in Eq. 7.13, and the limit $\lim_{t \rightarrow 0} \beta_2(t)$ converges by virtue of Th. 6.40. By virtue of the unitarity of ∇^0 , the connection $\nabla^{\det(\pi_! \nabla)}$ is a unitary connection, and the families index theorem (Th. 6.30) shows that it has curvature

$$\text{Curv}(\nabla^{\det(\pi_! \nabla)}) = (2\pi i)^{-n/2} \left[\pi_* \hat{A}(\Omega^\pi) \text{ch } \nabla \right]_{(2)}, \quad (7.15)$$

where $n = \dim X/Y$.

7.2.4 Summary

We have now produced a complex \mathbb{Z}_2 -graded line bundle $\det(\pi_! \nabla) \rightarrow Y$ with section (Def. 7.10), metric (defined by Eq. 7.11) and connection $\nabla^{\det(\pi_! \nabla)}$ (defined by Eq. 7.14) associated to the data of a Riemannian family with compact and spin fibres $\pi : X \rightarrow Y$, and a finite dimensional, complex and hermitian \mathbb{Z}_2 -graded vector bundle $E \rightarrow X$ with unitary superconnection ∇ . The curvature of the line bundle is computed by right hand side of the families index theorem (Eq. 7.15). Furthermore, it is easy to see that the construction is natural in the sense that commutes with pullbacks.

7.3 The holonomy of the determinant line bundle

We now calculate the holonomy of the determinant line bundle.

Theorem 7.16. *Let $X \rightarrow Y$ be a Riemannian family with compact and spin fibres, and $E \rightarrow X$ be a finite dimensional, complex, and hermitian \mathbb{Z}_2 -graded vector bundle with superconnection ∇ . The holonomy of the associated de-*

terminant line bundle is computed by the adiabatic limit of the τ -invariant. More precisely, let $\gamma : [0, 1] \rightarrow Y$ be a smooth loop (i.e. the map and all of its derivatives agree at the end points). Then

$$\text{Hol}_\gamma \det(\pi_! \nabla) = \text{a-lim } \tau_{\pi^{-1}\gamma}^{-1}(\nabla),$$

where the τ -invariant may be computed with any choice of metric on the base.

Proof. We will construct a mapping cylinder relating our situation to the analogous situation where the superconnection is an ordinary connection, and then use the theorem of Bismut and Freed [14] for the holonomy of determinant line bundles of families of Dirac operators coupled to connections.

Without loss of generality, we may restrict our attention to Riemannian maps $\pi : X \rightarrow S^1$, with the standard parameter θ and metric on S^1 . We endow the circle with the bounding spin structure. We form the associated family $\tilde{\pi} : X \times [0, 1] \rightarrow S^1 \times [0, 1]$, with the \mathbb{Z}_2 -graded vector bundle $E \times [0, 1] \rightarrow X \times [0, 1]$. Writing t as the parameter in $[0, 1]$, we endow $E \times [0, 1]$ with the superconnection $\tilde{\nabla} = (1 - t)\nabla + t\nabla$, where we have written $\nabla = \nabla + \omega$. By Stokes' theorem

$$2\pi i \int_{S^1 \times [0, 1]} \text{Curv}(\nabla^{\det(\pi_! \nabla)}) = \log \text{Hol}_{S^1} \det(\pi_! \nabla) - \log \text{Hol}_{S^1} \det(\pi_! \nabla).$$

We recall

$$\int_{S^1 \times [0, 1]} \text{Curv}(\nabla^{\det(\pi_! \nabla)}) = (2\pi i)^{-n/2} \int_{S^1 \times [0, 1]} \pi_* \hat{A}(\Omega^{\tilde{\pi}}) \text{ch } \nabla.$$

On the other hand, applying the APS theorem we see that

$$(2\pi i)^{-1} (\log \tau_X(\nabla) - \log \tau_X(\nabla)) \\ + (2\pi i)^{-n/2} \int_{X \times [0,1]} \hat{A}(\Omega^{X \times [0,1]}) \text{ch } \nabla = 0 \pmod{1}.$$

But

$$\begin{aligned} \text{a-lim} \int_{X \times [0,1]} \hat{A}(\Omega^{X \times [0,1]}) \text{ch } \nabla &= \int_{S^1 \times [0,1]} \hat{A}(\Omega^{S^1 \times [0,1]}) \pi_* \hat{A}(\Omega^{Y \times [0,1]}) \text{ch } \nabla \\ &= \int_{S^1 \times [0,1]} \pi_* \hat{A}(\Omega^{Y \times [0,1]}) \text{ch } \nabla. \end{aligned}$$

But then

$$(\text{Hol}_{S^1} \nabla) (\text{a-lim } \tau_X(\nabla)) = (\text{Hol}_{S^1} \nabla) (\text{a-lim } \tau_X(\nabla)).$$

Applying the theorem of Bismut and Freed [14] yields the result. \square

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Vita

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